

Spacetime dynamics of spinning particles — exact gravito-electromagnetic analogies

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We compare the rigorous equations describing the motion of spinning test particles in gravitational and electromagnetic fields, and show that if the Mathisson-Pirani spin condition holds then exact gravito-electromagnetic analogies emerge. These analogies provide a familiar formalism to treat gravitational problems, as well as a means for a comparison of the two interactions. Fundamental differences are manifest in the symmetries and time projections of the electromagnetic and gravitational tidal tensors. The physical consequences of the symmetries of the tidal tensors are explored comparing the following analogous setups: magnetic dipoles in the field of non-spinning/spinning charges, and gyroscopes in the Schwarzschild, Kerr, and Kerr-de Sitter spacetimes. The implications of the time-projections of the tidal tensors are illustrated by the work done on the particle in various frames; in particular, a reciprocity is found to exist: in a frame comoving with the particle, the electromagnetic (but not the gravitational) field does work on it, causing a variation of its proper mass; conversely, for “static observers”, a stationary gravitomagnetic (but not a magnetic) field does work on the particle, leading to a quantitative explanation of the Hawking-Wald spin interaction energy. The issue of hidden momentum, and its counterintuitive dynamical implications, is also analyzed. Finally, a number of issues regarding the electromagnetic interaction are clarified, namely the differences in the dynamics of electric and magnetic dipoles, and the physical meaning of Dixon’s equations.

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I. INTRODUCTION

The analogies between the equations of motion for gyroscopes in a gravitational field and magnetic dipoles in an electromagnetic field have been known for a long time, and were presented in many different forms throughout the years. This is the case for both the force equation (center of mass motion) and the spin evolution equation of these test particles in external fields. The former was first found by Wald [1] in the framework of linearized theory, who showed that the gravitational force exerted on a spinning pole-dipole test particle (hereafter a gyroscope), whose center of mass is *at rest* in a *stationary field*, takes the form $\vec{F}_G = K \nabla(\vec{H} \cdot \vec{S})$, where \vec{H} is the so-called “gravitomagnetic field”, K is some constant (depending on the precise definition of \vec{H} , e.g. [2–4]), and \vec{S} is the particle’s angular momentum. This formula is similar to the formula for the electromagnetic force on a magnetic dipole, $\vec{F}_{EM} = \nabla(\vec{B} \cdot \vec{\mu})$, where \vec{B} is the magnetic field and $\vec{\mu}$ is the dipole’s magnetic moment. The analogy was later cast in an *exact* form by one of the authors in [5], using

the *exact* gravitoelectromagnetic (GEM) inertial fields from the so-called 1+3 “quasi-Maxwell” formalism. The force was seen therein to consist of an electromagnetic-like term in the form above plus a term interpreted as the weight of the energy of the gravitomagnetic dipole, and the limit of validity of the analogy was extended to arbitrarily strong *stationary* fields *and* when the gyroscope’s worldline *is tangent to any time-like Killing vector field* (which comprehends e.g. circular trajectories with arbitrary speed in axisymmetric spacetimes). Recently it was shown that there is actually an exact, *covariant* and *fully general* analogy relating the two forces; this analogy is made explicit *not* in the framework of the “gravitoelectromagnetic” *vector* fields, but by using instead the tidal tensors of both theories, introduced in [6].

The analogy between the so-called “precession” of a gyroscope in a gravitational field and the precession of a magnetic dipole under the action of a magnetic field was noticed long ago, in the framework of linearized theory, by a number of authors [3, 4, 7, 8], who pointed out that the spin vector of a gyroscope at rest in a stationary field evolves as $d\vec{S}/dt = K\vec{S} \times \vec{H}$. This formula is similar to the formula for the precession of a magnetic dipole in a magnetic field, $d\vec{S}/dt = \vec{\mu} \times \vec{B}$. The analogy was later cast in an *exact* form in the framework of the GEM inertial fields, e.g. [4, 9, 10]; it is not covariant, holding only in a specific frame comoving with the particle, but, in the formulation in [9, 10] (as we shall see in detail), the test particle can be moving with arbitrary velocity in an *arbitrary field*.

These analogies provide a familiar formalism to treat otherwise complicated gravitational effects, as well as a means to compare the two interactions. In this work we explore these analogies, exemplifying their usefulness in some applications, and especially the insight they provide into fundamental, yet not well known, aspects of both interactions.

We will also make use of a third exact gravito-electromagnetic analogy (see e.g. [11–13]), this one a *purely* formal one (see [14]), relating the quadratic scalar invariants of the Maxwell and Weyl tensors [11, 12, 15]. The insight provided by this analogy will be crucial in the physical applications of Sec. IV B.

A. The equations of motion

In this work we start, in Sec. II, by writing the rigorous, general relativistic equations describing the motion of spinning test particles with gravitational and electromagnetic pole-dipole moments, subject to gravitational and electromagnetic external fields, in terms of physically meaningful quantities. This turns out not to be a straightforward task, as the covariant equations for this problem are still not generally well understood, with different methods and derivations leading to different versions of the equations, the relationship between them not being at all clear. Perhaps more surprising is the

fact that it is the electromagnetic (not the gravitational) field that has been posing more difficulties, with a number of misconceptions arising in the physical interpretation of the quantities involved in the equations (some authors [16, 17] have even concluded that such a covariant description is not possible). The main difficulties concern the interpretation of the time components of the forces and the mass of the test particles, and the fact that in most works, e.g. [18–23], equations of motion are presented (Eqs. (A2)-(A3)) which are symmetric in terms of electric and magnetic dipoles; that would, given the different nature of the two dipole models, seemingly lead (in many ways) to physically inconsistent predictions. This is discussed in detail in Appendix A, where it is shown that those equations are correct and consistent with the equations used in this work, Eqs. (8)-(9); the difficulties arise in the physical interpretation of the quantities appearing therein.

In order to form a determined system, the equations of motion need to be supplemented by a spin condition; the latter is even today still regarded as an open question, with a long history of debates concerning which one is the “best” condition (see Ref. [24] for a comprehensive review and list of references). In Sec. II A and Appendix C we briefly discuss its meaning, and the related problem of the relativistic definition of center of mass; a detailed discussion of the subtleties involved is given in the companion paper [25]. This subject is of relevance in this work because in order for the two physical gravito-electromagnetic analogies mentioned above (both for the force and for the precession of the spin vector) to be *exact*, the Mathisson-Pirani spin condition is required. This condition is poorly understood in most literature; namely, its degeneracy and the exotic helical solutions it allows. A lot of skepticism has been drawn into it because these helical solutions have been deemed unphysical. These assessments arise from misconceptions dissected in [25], where it is clarified that there is nothing wrong with Mathisson-Pirani spin condition; it is as valid as the Tulczyjew-Dixon [22, 26, 27], the Papapetrou-Corinaldesi [28], or any other physically reasonable condition (there are infinite possibilities), the choice between them being just a matter of convenience. Many applications in this work are examples where the Mathisson-Pirani condition is the best choice; that is discussed in detail in Appendix C.

Also related with the spin supplementary condition is an issue central to the understanding of the dynamics of a spinning particle: the decoupling of the 4-velocity from the 4-momentum, discussed in Sec. II D and Appendix D. The 4-velocity of a spinning particle is in general not parallel to its 4-momentum; it is said to possess “hidden momentum”, which, for a pole-dipole particle, consists of two parts: 1) the hidden momentum of electromagnetic origin and 2) a pure gauge term originated by the choice of spin condition, which we dub “inertial hidden momentum”; in the framework of the 1+3 formalism (with the appropriate condition), it is cast as the

exact inertial analogue of 1). The hidden momentum is known to lead to counter-intuitive behavior of the spinning particles; examples are the bobbings studied in [29], and also (as shown in [25]) Mathisson’s helical motions themselves, where a particle accelerates without the action of any force. In this work we present another, perhaps even more surprising consequence in Sec. IV A: due to the hidden momentum, a magnetic dipole with radial initial velocity in the field of a point charge accelerates in approximately *opposite* direction to the force.

B. The main realizations

Most of our applications, Secs. III-V of this paper, will deal with the tidal tensor formalism introduced in [6], and the exact analogy it unveils: both the electromagnetic force on a magnetic dipole and the gravitational force on a gyroscope are given by a contraction of a rank 2 magnetic type tidal tensor ($B_{\alpha\beta}$, $\mathbb{H}_{\alpha\beta}$), with the dipole/spin 4-vector. Here $B_{\alpha\beta}$ gives the tidal effects of the magnetic field and $\mathbb{H}_{\alpha\beta}$ is the magnetic part of the Riemann tensor, both measured in the particle’s rest frame. This makes this formalism especially suited to compare the two forces — it amounts to simply comparing the two tidal tensors. Such comparison is done through Einstein and Maxwell equations, as they also can be written in terms of tidal tensors. Apart from the non-linearity of $\mathbb{H}_{\alpha\beta}$, the tensorial structure differs essentially when the fields vary along the test particle’s worldline, since this endows $B_{\alpha\beta}$ with an antisymmetric part (non-vanishing even in vacuum), and a non-vanishing time projection along that worldline, whereas its gravitational counterpart is spatial and, *in vacuum*, symmetric. This signals fundamental differences in the interactions, as the antisymmetric parts of the electromagnetic tidal tensors (as well as their time-projections) encode the laws of electromagnetic induction. We discuss them in two separate sections.

In Sec. IV we explore the physical consequences of the different symmetries of the gravitational and electromagnetic tidal tensors. They are seen to imply e.g. that particles moving in a non-homogeneous electromagnetic field always measure a non-vanishing $B_{\alpha\beta}$ (thus feel a force), which is not necessarily the case in gravity. The following analogous setups are compared: magnetic dipoles in the field of non-spinning/spinning charges, and gyroscopes in the Schwarzschild, Kerr, and Kerr-dS spacetimes. It is seen that in the cases where $B_{\alpha\beta}$ reduces to $B_{[\alpha\beta]}$, we have $\mathbb{H}_{\alpha\beta} = 0$ in the gravitational analogue (i.e., gyroscopes feel no force). Geodesic motions for spinning particles are found to exist in Schwarzschild (radial geodesics) and in Kerr-dS (circular equatorial geodesics) spacetimes.

In Sec. V we explore the physical content of the time projections of the forces in different frames. These are related with the rate of work done on the test particle by the external fields; in order to obtain that relation-

ship, we start by deriving the general equation yielding the variation of energy of a particle with multipole structure with respect to an arbitrary congruence of observers. This generalizes the power equation in [9] for test particles possessing hidden momentum and varying mass, and non-spatial forces. We then show that the electromagnetic force has a non-vanishing projection along the particle's 4-velocity U^α , which is the power transferred to the dipole by Faraday's induction, reflected in a variation of its proper mass m . The projection of the gravitational force along U^α , by contrast, vanishes (since $\mathbb{H}_{\alpha\beta}$ is spatial with respect to U^α , manifesting the absence of an analogous induction effect), leading to the conservation of the gyroscope's mass. Also of particular interest in this context are the time projections as measured by "static observers", analyzed in Sec. VB. For these observers, the time projection of the electromagnetic force vanishes, meaning that the total work done on the magnetic dipole is zero. This reflects the well known fact that the work done by the magnetic field is zero, which in this framework is seen to arise from an exchange of energy between three forms: translational kinetic energy, proper mass m , and "hidden energy", which occurs in a way such that their variations cancel out, keeping the total energy constant. In the gravitational case, since the induction effects are absent, such cancellation does not occur and, unlike its electromagnetic counterpart, a stationary field *does work* on mass currents, there existing an associated potential spin-curvature potential energy. Considering, as a special case, the gyroscope to be a small Kerr black hole, falling axially into a large Kerr black hole, it yields Wald's [1] gravitational spin-spin interaction energy, which *quantitatively* explains the spin-dependence of Hawking's upper bound [30] for the energy released by gravitational radiation when two black holes collide.

It is worth noting that the above mentioned differences between the two interactions are *not* negligible in the weak field and slow motion regime, as is shown in detail in Sec. III (this is commonly overlooked in the literature concerning this regime).

C. Beyond the pole-dipole

In Sec. VI we go beyond the pole-dipole approximation, including the moments of quadrupole order, to clarify the mechanism by which the proper mass of a spinning particle in a electromagnetic field varies, and solve an apparent contradiction of the former approximation: On the one hand (as stated above), we show in Sec. V that the projection of the force on a magnetic dipole along its worldline is the rate of work done on it by the induced electric field \vec{E}_{ind} (as measured in the particle's frame), which is manifest in a variation of the particle's proper mass m . For a quasi-rigid spinning particle, the latter is a variation of rotational kinetic energy. Likewise, the angular momentum of the particle (and its magnitude

$S = \sqrt{S^2}$) must also vary, as expected from the fact that \vec{E}_{ind} , by having a curl, should torque the spinning body. On the other hand, however, no trace of that torque ($\vec{\tau}_{ind}$) is found to dipole order, where the torque on a magnetic dipole is the textbook expression $\vec{\tau} = \vec{\mu} \times \vec{B}$, leading to the conservation of S . In Sec. VIA this apparent paradox is clarified; indeed a torque $\vec{\tau}_{ind}$ generated by the induced electric field is exerted on the particle (and it is governed by the time projection of the magnetic tidal tensor $B^\alpha_\beta U^\beta$), but, it involves *second moments* of the charge (i.e., of quadrupole order), which is the reason why it does not show up in pole-dipole approximations; however the rate of work it does, $\vec{\tau}_{ind} \cdot \vec{\Omega}$ (and the corresponding variation of m), can be written in terms of the the magnetic dipole moment, yielding exactly the time projection of the dipole force along its worldline, as obtained in Sec. VA. Then in Sec. VIB we study the analogous gravitational problem, showing that, as expected (as $\mathbb{H}_{\alpha\beta}$ is spatial), no analogous torque exists. In particular it is shown that the gravitational field cannot torque a "spherical" body (case in which the electromagnetic torque reduces essentially to $\vec{\tau}_{ind}$), which is seen to lie at the heart of the frame dragging effect.

D. Notation and conventions

1. *Signature and signs.* We use the signature $-++$; $\epsilon_{\alpha\beta\sigma\gamma} \equiv \sqrt{-g}[\alpha\beta\gamma\delta]$ denotes the Levi-Civita tensor, and we follow the orientation $[1230] = 1$ (i.e., in flat spacetime, $\epsilon_{1230} = 1$). $\epsilon_{ijk} \equiv \epsilon_{ijk0}$ is the 3-D alternating tensor. We use the convention for the Riemann tensor: $R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \dots$
2. Sometimes we use the abbreviation $\epsilon_{\alpha\beta\gamma} \equiv \epsilon_{\alpha\beta\gamma\delta} U^\delta$, where U^α is the 4-velocity of the test particle's CM.
3. We use bold fonts to denote tensors \mathbf{T} (including 4-vectors \mathbf{v}), and arrows for 3-vectors \vec{v} . Greek letters $\alpha, \beta, \gamma, \dots$ denote 4-D spacetime indices, Roman letters i, j, k, \dots denote 3-D spatial indices. Following the usual practice, sometimes we use component notation $T^{\alpha\beta}$ to refer to a tensor \mathbf{T} .
4. By u^α we denote a generic unit time-like vector, which can be interpreted as the instantaneous 4-velocity of some observer. $U^\alpha \equiv dz^\alpha/d\tau$ is the tangent vector to the body's representative worldline $z^\alpha(\tau)$, which in this work is taken to be a suitably defined center of mass; U^α is thus the 4-velocity of the test particle's center of mass.
5. *Time and space projectors.* $(\top^u)^\alpha_\beta \equiv -u^\alpha u_\beta$ and $(h^u)^\alpha_\beta \equiv \delta^\alpha_\beta + u^\alpha u_\beta$ denote, respectively, the projectors parallel and orthogonal to a unit time-like vector u^α ; may be interpreted as the time and space projectors in the local rest frame of an observer of 4-velocity u^α .

6. *Tensors resulting from a measurement process.* $(A^u)^{\alpha_1 \dots \alpha_n}$ denotes the tensor \mathbf{A} as measured by an observer $\mathcal{O}(u)$ of 4-velocity u^α . For example, $(E^u)^\alpha \equiv F^\alpha_\beta u^\beta$, $(E^u)_{\alpha\beta} \equiv F_{\alpha\gamma;\beta} u^\gamma$ and $(\mathbb{E}^u)_{\alpha\beta} \equiv R_{\alpha\nu\beta\nu} u^\nu u^\mu$ denote, respectively, the electric field, electric tidal tensor, and gravito-electric tidal tensor as measured by $\mathcal{O}(u)$. Analogous forms apply to their magnetic/gravitomagnetic counterparts.

For 3-vectors we use notation $\vec{A}(u)$; for example, $\vec{E}(u)$ denotes the electric 3-vector field as measured by $\mathcal{O}(u)$ (i.e., the space part of $(E^u)^\alpha$, written in a frame where $u^i = 0$). When $u^\alpha = U^\alpha$ (i.e., the particle's CM 4-velocity) we drop the superscript (e.g. $(E^U)^\alpha \equiv E^\alpha$), or the argument of the 3-vector: $\vec{E}(U) \equiv \vec{E}$.

7. *Electromagnetic field.* The Maxwell tensor $F^{\alpha\beta}$ and its dual $\star F^{\alpha\beta}$ decompose in terms of the electric $(E^u)^\alpha \equiv F^\alpha_\beta u^\beta$ and magnetic $(B^u)^\alpha \equiv \star F^\alpha_\beta u^\beta$ fields measured by an observer of 4-velocity u^α as

$$F_{\alpha\beta} = 2u_{[\alpha}(E^u)_{\beta]} + \epsilon_{\alpha\beta\gamma\delta} u^\delta (B^u)^\mu ; \quad (1)$$

$$\star F_{\alpha\beta} = 2u_{[\alpha}(B^u)_{\beta]} - \epsilon_{\alpha\beta\gamma\sigma} u^\sigma (E^u)^\gamma . \quad (2)$$

8. *Static observers.* In stationary, asymptotically flat spacetimes, we dub “static observers” the *rigid* congruence of observers whose worldlines are tangent to the temporal Killing vector field $\xi = \partial/\partial t$; may be interpreted as the set of points rigidly fixed to the “distant stars” (the asymptotic inertial rest frame of the source). For the case of Kerr spacetime, these correspond to the observers of zero 3-velocity in Boyer-Lindquist coordinates. This agrees with the convention in e.g. [31, 32]. Note however that the denomination “static observers” is employed with a different meaning in some literature, e.g. [33], where it designates rigid, *vorticity-free* congruences (existing only in *static spacetimes*). In the case of the electromagnetic systems in flat spacetimes, by static observers we mean the globally inertial rest frame of the sources.

9. *GEM.* This is the acronym for “gravitoelectromagnetism”. By “inertial GEM fields”, we mean the fields of inertial forces that arise from the 1+3 splitting of spacetime: the gravitoelectric field \vec{G} , which plays in this framework a role analogous to the electric field of electromagnetism, and the gravitomagnetic field \vec{H} , analogous to the magnetic field. These fields are discussed in detail in the companion paper [34].

II. EQUATIONS OF MOTION FOR SPINNING PARTICLES IN A CURVED AND ELECTROMAGNETIC BACKGROUND

In most of this work we will be dealing with the dynamics of the so-called pole-dipole spinning test parti-

cles. We consider systems composed of a test body plus background gravitational and electromagnetic fields. Let $(T_{\text{tot}})^{\alpha\beta} = \Theta^{\alpha\beta} + (T_{\text{matter}})^{\alpha\beta}$ denote the total energy-momentum tensor, which splits into the electromagnetic stress-energy tensor $\Theta^{\alpha\beta}$ and the energy-momentum of the matter $(T_{\text{matter}})^{\alpha\beta}$. Moreover, let $T^{\alpha\beta}$ and j^α denote, respectively, the energy momentum tensor and the current density 4-vector of the test body. We also consider that the only matter and currents present are the ones arising from the test body: $(T_{\text{matter}})^{\alpha\beta} = T^{\alpha\beta}$, $(j_{\text{tot}})^\alpha = j^\alpha$. In this case (see Appendix D for details) the conservation of total energy-momentum tensor yields:

$$(T_{\text{tot}})^{\alpha\beta}_{;\beta} = 0 \Rightarrow T^{\alpha\beta}_{;\beta} = -\Theta^{\alpha\beta}_{;\beta} \Leftrightarrow T^{\alpha\beta}_{;\beta} = F^{\alpha\beta} j_\beta , \quad (3)$$

where $F^{\alpha\beta}$ is the Maxwell tensor of the *external* (background) electromagnetic field.

In a multipole expansion the body is represented by the moments of j^α (its “electromagnetic skeleton”) and a set of moments of $T^{\alpha\beta}$, called “inertial” or “gravitational” moments (forming the so called [35] “gravitational skeleton”). Truncating the expansion at dipole order, the equations of motion for such a particle involve only two moments of $T^{\alpha\beta}$:

$$P^\alpha \equiv \int_{\Sigma(\tau,U)} T^{\alpha\beta} d\Sigma_\beta , \quad (4)$$

$$S^{\alpha\beta} \equiv 2 \int_{\Sigma(\tau,U)} r^{[\alpha} T^{\beta]\gamma} d\Sigma_\gamma , \quad (5)$$

and two electromagnetic moments [21]:

$$q \equiv \int_{\Sigma} j^\alpha d\Sigma_\alpha , \quad (6)$$

$$Q^{\alpha\beta} \equiv \int_{\Sigma(\tau,U)} (r^{[\alpha} j^{\beta]} U^\gamma - r^{[\alpha} U^{\beta]} j^\gamma) d\Sigma_\gamma . \quad (7)$$

Here the motion of the test particle is described by a reference worldline $z^\alpha(\tau)$ relative to which the moments are taken, where τ is the proper time. The precise choice of this worldline will be discussed below. $U^\alpha \equiv dz^\alpha/d\tau$ denotes its (unit) tangent vector; $P^\alpha(\tau)$ is the 4-momentum of the test particle; $S^{\alpha\beta}(\tau)$ is the angular momentum about the point $z^\alpha(\tau)$ of the reference worldline; $r^\alpha(\tau) \equiv x^\alpha - z^\alpha(\tau)$, where $\{x^\alpha\}$ is a chart on spacetime; $\Sigma(\tau, U) \equiv \Sigma(z(\tau), U)$ is the spacelike hypersurface generated by all geodesics orthogonal to U^α at the point $z^\alpha(\tau)$; $d\Sigma_\gamma \equiv n_\gamma d\Sigma$, where n_γ is the (past-pointing) unit normal to $\Sigma(\tau, u)$ and $d\Sigma$ is the 3-volume element on $\Sigma(\tau, u)$; q denotes the total charge, which is an invariant (making Σ in this case arbitrary); and $Q^{\alpha\beta}$ is the electromagnetic dipole moment about the point $z^\alpha(\tau)$. Expressions (4), (5) and (7) are integrals of rank 2 and 3 tensors over Σ (i.e., amount to adding tensor components at different points), which requires a justification: the pole-dipole approximation amounts to considering $T^{\alpha\beta}$ and j^α non-vanishing only in a very small region around $z^\alpha(\tau)$, so that only terms linear in r are

kept; since to first order spacetime can always be taken as flat, these integrals are meaningful mathematical operations and do indeed define tensors, just like in flat spacetime.

The equations of motion are [21, 27, 36] (see Appendix A for a discussion):

$$\frac{DP^\alpha}{d\tau} = qF^\alpha_\beta U^\beta + \frac{1}{2}F^{\mu\nu;\alpha}\mu_{\mu\nu} - \frac{1}{2}R^\alpha_{\beta\mu\nu}S^{\mu\nu}U^\beta + E^\alpha_\beta d^\beta + F^\alpha_\beta \frac{Dd^\beta}{d\tau}, \quad (8)$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]} + 2\mu^{\theta[\beta}F^{\alpha]}_\theta + 2d^{[\alpha}F^{\beta]}_\gamma U^\gamma, \quad (9)$$

where $F^{\alpha\beta}$ is the background Maxwell tensor, d^α and $\mu_{\alpha\beta}$ are the time and space projections of $Q^{\alpha\beta}$ with respect to U^α :

$$d^\alpha = -Q^{\alpha\beta}U_\beta; \quad \mu^{\alpha\beta} \equiv (h^U)^\alpha_\gamma (h^U)^\beta_\delta Q^{\gamma\delta}, \quad (10)$$

where $(h^U)^\beta_\alpha \equiv \delta^\beta_\alpha + U^\beta U_\alpha$. Introducing also μ^α by

$$\mu_{\alpha\beta} \equiv \epsilon_{\alpha\beta\gamma\delta}\mu^\gamma U^\delta; \quad \mu^\alpha = \frac{1}{2}\epsilon^\alpha_{\beta\gamma\delta}U^\beta\mu^{\gamma\delta}, \quad (11)$$

the vectors d^α and μ^α have the form $(0, \vec{d})$ and $(0, \vec{\mu})$ in the frame where $U^i = 0$, and correspond to the usual notion of electric and magnetic dipole moments with respect to an observer of 4-velocity U^α . With respect to an arbitrary observer of 4-velocity u^α we can decompose:

$$Q^{\alpha\beta} = 2(d^u)^{[\alpha}u^{\beta]} + \epsilon^{\alpha\beta\gamma\delta}(\mu^u)_\gamma u_\delta, \quad (12)$$

$(d^u)^\alpha$ and $(\mu^u)^\alpha = \frac{1}{2}\epsilon^\alpha_{\beta\gamma\delta}u^\beta\mu^{\gamma\delta}$ being the electric and magnetic dipole moments as measured by u^α . In some applications we will assume $\mu^{\alpha\beta}$ to be proportional to the spin tensor: $\mu^{\alpha\beta} = \sigma S^{\alpha\beta}$, where σ is the gyromagnetic ratio.

The first term in (8) is the Lorentz force; the second term, $\frac{1}{2}F^{\mu\nu;\alpha}\mu_{\mu\nu} \equiv F^\alpha_{\text{EM}}$, is the force due to the tidal coupling of the electromagnetic field to the magnetic dipole moment; and the third, $-\frac{1}{2}R^\alpha_{\beta\mu\nu}S^{\mu\nu} \equiv F^\alpha_G$ is the Mathisson-Papapetrou spin-curvature force. The last two terms are the force exerted on the electric dipole, consisting of a tidal term $F^\alpha_{\gamma;\beta}U^\gamma d^\beta$ and of a *non tidal* term $F^\alpha_\beta Dd^\beta/d\tau$.

Up until now, the reference worldline $z^\alpha(\tau)$, relative to which the moments in Eqs. (8)-(9) above are taken, is still undefined. Had we made an exact expansion keeping all the infinite multipole moments as in [37, 38], such worldline would be *arbitrary*. Herein, however, it must be assumed that it passes through the body (or close enough), so that the pole-dipole approximation is valid; it will be chosen as being prescribed by a suitably defined center of mass of the test particle. As discussed in the next section, that is done through a supplementary condition $S^{\alpha\beta}u_\beta = 0$, for some time-like unit vector field u^α . If $F^{\alpha\beta} = 0$ there are 13 unknowns in Eqs. (8)-(9) (P^α , 3

independent components of U^α , and 6 independent components of $S^{\alpha\beta}$) for only 10 equations. The condition $S^{\alpha\beta}u_\beta = 0$, for a *definite* u^α , closes the system as it kills off 3 components of the angular momentum. But in the general case $F^{\alpha\beta} \neq 0$ one needs also to give the laws of evolution of $Q^{\alpha\beta}$ in order for the system to be determined, cf. [39].

A. Center of mass and spin supplementary condition

In relativistic physics, the center of mass of a spinning particle is observer¹ dependent. This is illustrated in Fig. 1 of [25]. Thus one needs to specify the frame in which the center of mass is to be evaluated. That amounts to supplement Eqs. (8)-(9) (which, as discussed above, would otherwise be undetermined) by the spin subsidiary condition $S^{\alpha\beta}u_\beta = 0$, as we will show next. The vector

$$(d^u_G)^\alpha \equiv -S^{\alpha\beta}u_\beta \quad (13)$$

yields the “mass dipole moment” (i.e. the mass times the displacement of the reference worldline relative to the center of mass) as measured by the observer \mathcal{O} of 4-velocity u^α . In order to see this let us, for simplicity, take $F^{\alpha\beta} = 0$ so that the integral (5) does not depend (to pole-dipole order) on Σ , by virtue of the conservation equations $T^{\alpha\beta}_{;\beta} = 0$. And let us set up, at the reference worldline (cf. [40]), a locally nearly Lorentz frame \hat{e}_α momentarily comoving with \mathcal{O} ; i.e., $\hat{e}_0 = u^\alpha$ and the triad \hat{e}_i spans the instantaneous rest space of \mathcal{O} . In this frame, $u^{\hat{i}} = 0$ and $S^{\hat{i}\hat{j}}u_{\hat{j}} = S^{\hat{i}\hat{0}}u_{\hat{0}}$; also $r^{\hat{\alpha}} = x^{\hat{\alpha}}$, and thus from Eq. (5) we have:

$$S^{\hat{i}\hat{0}} = 2 \int_{\Sigma(\tau, u)} x^{[\hat{i}}T^{\hat{0}]\gamma} d\Sigma_\gamma = \int x^{\hat{i}}T^{\hat{0}\hat{0}} d^3x \equiv m(u)x^{\hat{i}}_{\text{CM}}(u); \quad (14)$$

note that $r^{\hat{0}} = 0$, since the integration is performed in the hypersurface $\Sigma(\tau, u)$ orthogonal to u^α . Here $m(u) \equiv \int T^{\hat{0}\hat{0}} d^3x = -P^\alpha u_\alpha$ denotes the mass as measured by observer \mathcal{O} . We see that $S^{\hat{i}\hat{0}}$ is by definition the mass dipole in the frame \hat{e}_α : $S^{\hat{i}\hat{0}} = m(u)x^{\hat{i}}_{\text{CM}}(u) \equiv (d^u_G)^{\hat{i}}$,

¹ We will make reference to quantities such as the center of mass, mass dipole, etc as measured by a given *observer* $\mathcal{O}(u)$ with some 4-velocity u^α . This demands an explanation. These quantities involve integrals over a spacelike 3-D hypersurface Σ ; Σ is defined as the hypersurface generated by all geodesics orthogonal to the observer’s worldline; *in flat spacetime* this corresponds to the notion of rest space of $\mathcal{O}(u)$, defined as the 3-space orthogonal to the vector field obtained by parallel transporting u^α to every point. In any case it is the 4-velocity u^α , at the given point, that entirely defines the hypersurface of integration, hence we refer to this as the quantities measured by a given *observer* of 4-velocity u^α , in place of quantities measured in a given *frame* (e.g. the rest frame of u^α).

and $x_{\text{CM}}^i(u) = S^{i0}/m(u)$ is the center of mass position. Thus the condition $S^{\alpha\beta}u_\beta = 0$, implying in this frame $S^{i0} = 0 \Rightarrow x_{\text{CM}}^i(u) = 0$, is precisely the condition that the reference worldline is the center of mass as measured in this frame (or, equivalently, that the “mass dipole moment” vanishes for such an observer). The condition $S^{\alpha\beta}u_\beta = 0$ allows us to write $S^{\alpha\beta} = \epsilon^{\alpha\beta}_{\gamma\delta} S^\gamma u^\delta$, where S^α is the spin 4-vector, defined as the 4-vector with components $(0, \vec{S})$ in the rest frame of \mathcal{O} (where \vec{S} is the angular momentum vector measured by this observer). For details on how the center of mass position changes in a change of observer, we refer the reader to [25].

In general one wants the equations of motion not to depend on quantities (the center of mass) measured by a particular observer, but instead on a center of mass defined only in terms of “intrinsic” properties of the particle. Two conditions for accomplishing this are common in the literature²: the Frenkel-Mathisson-Pirani condition $S^{\alpha\beta}U_\beta = 0$ (hereafter Pirani’s condition, as it is best known) and the Möller-Tulczyjew-Dixon condition $S^{\alpha\beta}P_\beta = 0$ (hereafter Dixon’s condition). The former, which is analogous to the magnetic side condition $\mu^{\alpha\beta}U_\beta = 0$ (if we assume $\mu^{\alpha\beta} = \sigma S^{\alpha\beta}$, they are the same), is the one on which the *exact* gravito-electromagnetic analogies presented in this work rely. It seems also the most natural choice, as it amounts to computing the center of mass in its proper frame, i.e., *in the frame where it has zero 3-velocity*. Such center of mass is dubbed a “proper center of mass” [41]. This condition arises also in a natural fashion in a number of treatments [42, 43] (see also [44]); it turns out, however, that it does not determine the worldline uniquely. For instance, in the special case of Minkowski spacetime in the absence of electromagnetic fields, Eqs. (8)-(9), supplemented by $S^{\alpha\beta}U_\beta = 0$, are known to lead, in addition to the expected straightline motion, to an infinite set of helical motions, which were notably found by Mathisson [45]. These solutions have been deemed unphysical by some authors, with the argument that the radius of the helices can be arbitrarily large (see e.g. [27, 36, 46, 47]), which would be contradicted by experiment. As shown in detail in [25], those statements originate from a misconception in the derivation of the radius. The reason for the indeterminacy of this condition is seen from the fact that for a spinning particle not only are there infinite possible centers of mass, depending on the observer, but there are also infinite *proper* centers of mass [41]; the motion of the latter yielding the helices. Their radius is finite and contained within a tube of radius $R_{\text{max}} = S/M$, which is actually the *minimum* size that a classical spinning particle can have, if it is to have finite S and positive mass without violating the speed of light limit [48] (see also [41, 49] for a different argument). The different world-

lines compatible with this condition are just equivalent descriptions of the same motion.

The Dixon condition $S^{\alpha\beta}P_\beta = 0$ amounts to computing the CM *in the frame where it has zero 3-momentum*. This condition determines uniquely the center of mass worldline [22, 50, 51]; there is no ambiguity in this case, since P^α is given in advance by (4), and for this reason is preferred by a number of authors. Note that in flat spacetime the worldline specified by $S^{\alpha\beta}P_\beta = 0$ corresponds to Mathisson’s non-oscillatory solution; but in the presence of gravitational/electromagnetic field, P^α cannot in general parallel to U^α if we assume $S^{\alpha\beta}P_\beta = 0$ (see Eq. (35) of [29] for a general expression for U^α in terms of P^α and $S^{\alpha\beta}$). In the presence of an electromagnetic field it presents some counter-intuitive properties as well (see Appendix C 2).

The fundamental point that we want to emphasize here is that (contrary to the point of view in some literature) these two conditions, as well as others explored in [40] (including non-covariant ones such as the Corinaldesi-Papapetrou condition [28]), are equivalent descriptions of the motion of the test particle, the choice between them being a matter of convenience. This is one of the main points made in [25]. It is clear that for a free particle in flat spacetime the Dixon condition gives the simplest, non-degenerate, description (straightline motion); but in some more complicated systems the Pirani condition turns to be more suitable (examples are the applications in Secs. IV A, IV B and V B; this is discussed in detail in Appendix C). In the case of massless particles, it has been argued in [52, 53] that the latter condition is actually the only one that can be applied.

In the present work we will be mainly using Pirani’s condition, since, as we shall see, it is the one on which the *exact* gravito-electromagnetic analogies rely. The differences between the two conditions, as well as their impact on the gravito-electromagnetic analogies, are discussed in Appendix C. With this choice, we have $S^{\mu\nu} = \epsilon^{\mu\nu\tau\lambda} S_\tau U_\lambda$, where S^α is the spin 4-vector, defined as being the 4-vector with components $(0, \vec{S})$ in the CM frame. Substituting in Eqs. (8)-(9) (and also performing the contractions with U^α) we obtain

$$\begin{aligned} \frac{DP^\alpha}{d\tau} &= qE^\alpha + E^{\alpha\beta}d_\beta + B^{\beta\alpha}\mu_\beta - \mathbb{H}^{\beta\alpha}S_\beta \\ &\quad + F^\alpha_\beta \frac{Dd^\beta}{d\tau}; \end{aligned} \quad (15)$$

$$\frac{D_F S_\mu}{d\tau} = \epsilon_{\mu\alpha\beta\nu} U^\nu (d^\alpha E^\beta + \mu^\alpha B^\beta), \quad (16)$$

where $D_F/d\tau$ denotes the Fermi-Walker derivative, $E^\alpha \equiv F^{\alpha\beta}U_\beta$ and $B^\alpha \equiv \star F^{\alpha\beta}U_\beta$ are the electric and magnetic fields as measured by the test particle, and $E_{\alpha\beta} \equiv F_{\alpha\mu;\beta}U^\mu$, $B_{\alpha\beta} \equiv \star F_{\alpha\mu;\beta}U^\mu$ and $\mathbb{H}_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\sigma}U^\mu U^\sigma$ are, respectively, the electric, magnetic and gravitomagnetic tidal tensors as defined in [6, 34], *measured by the test particle*.

² A review (with a comprehensive list of references) on the literature regarding this subject may be found in [24].

B. Force on gyroscope vs. force on magnetic dipole - exact analogy based on tidal tensors

Herein we are interested in purely magnetic dipoles, i.e., dipoles whose electric moment vanishes in the CM frame; this is ensured by the condition $d^\alpha = 0 \Rightarrow Q^{\alpha\beta} = \mu^{\alpha\beta}$, see expressions (10). In this case Eq. (15) simplifies to:

$$\frac{DP^\alpha}{d\tau} = qF^{\alpha\beta}U_\beta + B^{\beta\alpha}\mu_\beta - \mathbb{H}^{\beta\alpha}S_\beta. \quad (17)$$

These equations manifest the physical analogy $B_{\alpha\beta} \leftrightarrow \mathbb{H}_{\alpha\beta}$, summarized in Table I: (i) both the electromagnetic force on a magnetic dipole and the gravitational force on a gyroscope are determined by a contraction of the spin/magnetic dipole 4-vector with a magnetic type tidal tensor. $B_{\alpha\beta}$ is a covariant derivative, keeping U^α fixed (covariantly constant), of the magnetic field $B^\alpha = \star F^\alpha{}_\beta U^\beta$ measured by the test particle: $B_{\alpha\beta} = B_{\alpha;\beta}|_{U=\text{const.}}$; i.e., it is a derivative of B^α as measured in the *inertial frame momentarily comoving with the particle*. For this reason it is dubbed $B_{\alpha\beta}$ the *magnetic tidal tensor*, and its gravitational counterpart $\mathbb{H}_{\alpha\beta}$ the *gravitomagnetic tidal tensor* [6]. (ii) It turns out that $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ obey the formally similar equations (I.2) and (I.3) in Table I, which in one case are Maxwell equations, and in the other are *exact* Einstein equations. That is: the traces (I.2) are, respectively, the time projection of the electromagnetic Bianchi identity $\star F^{\alpha\beta}{}_{;\beta} = 0$ and the time-time projection of the algebraic Bianchi identities $\star R^{\gamma\alpha}{}_{\gamma\beta} = 0$; the antisymmetric parts (I.3a) are, respectively, the space projection of Maxwell's equations $F^{\alpha\beta}{}_{;\beta} = 4\pi j^\alpha$ and the time-space projection of Einstein's equations $R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\alpha{}_\alpha)$. The electromagnetic equations take a familiar form in an inertial frame: Eq. (I.2a) becomes $\nabla \cdot \vec{B} = 0$; the space part of (I.3a) is the Maxwell-Ampère law $\nabla \times \vec{B} = \partial \vec{E}/\partial t + 4\pi \vec{j}$. The latter means that the space part of $B_{[\alpha\beta]}$ encodes the curl of B^α , which is actually a more general statement, holding in arbitrarily accelerated frames; in such frames [assumed non-rotating and non-shearing, cf. Eq. (93) below] $U_{\alpha;\beta} = -a_\alpha U_\beta$, and we have:

$$\epsilon^{\beta\gamma}{}_{\alpha\delta} B_{\gamma\beta} U^\delta = \epsilon^{\beta\gamma}{}_{\alpha\delta} B_{\gamma;\beta} U^\delta \Rightarrow \epsilon^{ikj} B_{jk} = (\nabla \times \vec{B})^i. \quad (18)$$

Expressing also the second member of (I.3a) in terms of the electric and magnetic fields E^α and B^α measured in this frame, we obtain, in 3-vector notation,

$$\nabla \times \vec{B} = \frac{D\vec{E}}{d\tau} - \vec{a} \times \vec{E} + 4\pi \vec{j}$$

which is the generalization of Maxwell-Ampère law for accelerated frames (cf. Eq. (19) of [13]). This equation, as well as Eq. (21) below, is of use if one wishes to express Eqs. (I.3a) and (19a) in terms of the fields measured in the particle's CM frame (as it in general accelerates).

According to Table I, both in the case of the electromagnetic force on a magnetic dipole, and the case of the gravitational force on a gyroscope, it is the magnetic tidal tensor, *as seen by the test particle* of 4-velocity U^α , that determines the force exerted upon it. The explicit analogy in Table I is thus ideally suited to compare the two forces, because in this framework it amounts to comparing $B_{\alpha\beta}$ to $\mathbb{H}_{\alpha\beta}$. The most important differences between them are: i) $B_{\alpha\beta}$ is linear in the electromagnetic potentials and vector fields, whereas $\mathbb{H}_{\alpha\beta}$ is not linear in the metric tensor, nor in the GEM “vector” fields; for a detailed discussion of this aspect, we refer the reader to the companion paper [34]; ii) *in vacuum*, $\mathbb{H}_{[\alpha\beta]} = 0$ (symmetric tensor), whereas $B_{[\alpha\beta]} = \frac{1}{2}\star F_{\alpha\beta;\gamma}U^\gamma \neq 0$ (generically not symmetric, even in vacuum); iii) time components: $\mathbb{H}_{\alpha\beta}$ is a spatial tensor, whereas $B_{\alpha\beta}$ is not. These last two differences, which are clear from equations (I.3)-(I.4), are the ones in which we are mostly interested in the present work. In what follows we explore some physical consequences.

Note this important aspect of Eq. (I.3a), considering for simplicity the vacuum case $j^\alpha = 0$: it tells us that when the field $F_{\alpha\beta}$ varies along the worldline of the observer U^α , that endows $B_{\alpha\beta}$ with an antisymmetric part, meaning also that the tensor itself is non-vanishing. Now, since $B_{\alpha\beta}$ in the force (I.1a) is the magnetic tidal tensor *as measured by the particle* (i.e., U^α is the test particle's 4-velocity), this means that whenever the particle moves in a non-homogeneous field, a force will be exerted on it (except possibly for very special orientations of $\vec{\mu}$). In the inertial frame momentarily comoving with the particle, this can be interpreted as the varying electric field measured in this frame generating, via the induction law $\nabla \times \vec{B} = \partial \vec{E}/\partial t$, a curl in the magnetic field \vec{B} , implying that the particle sees a non-vanishing magnetic tidal tensor. The fact that its gravitomagnetic counterpart $\mathbb{H}_{\alpha\beta}$ is symmetric in vacuum tells us that no analogous induction phenomenon occurs in gravity; this observation is crucial in some applications in this work (see Sec. IV).

There is an electric counterpart to this analogy, relating the electric tidal tensor $E_{\alpha\beta}$ with the electric part of the Riemann tensor:

$$E_{\alpha\beta} \equiv F_{\alpha\mu;\beta}U^\mu \longleftrightarrow \mathbb{E}_{\alpha\beta} \equiv R_{\alpha\mu\beta\nu}U^\mu U^\nu,$$

which is manifest in the worldline deviations of both theories, see [6], and together they form the gravito-electromagnetic analogy based on tidal tensors [6, 34]. These tensors obey the following equations, which will be useful in this work:

$$E_{[\alpha\beta]} = \frac{1}{2}F_{\alpha\beta;\gamma}U^\gamma \quad (\text{a}); \quad \mathbb{E}_{[\alpha\beta]} = 0 \quad (\text{b}) \quad (19)$$

Eq. (19a) results from the space projection of the identity $\star F^{\alpha\beta}{}_{;\beta} = 0$; and Eq. (19b) from the time-space projection of the identity $\star R^{\gamma\alpha}{}_{\gamma\beta} = 0$. Contracting (19a) with the spatial 3-form $\epsilon_{\alpha\beta\gamma\delta}U^\delta$ yields Eq. (I.4a) of Table I. Again, for a (non-rotating and non-shearing) arbitrarily

Table I: Analogy between the electromagnetic force on a magnetic dipole and the gravitational force on a gyroscope

Electromagnetic Force on a Magnetic Dipole	Gravitational Force on a Spinning Particle
$F_{EM}^\beta = B_\alpha^\beta \mu^\alpha; \quad B_\beta^\alpha \equiv \star F_{\mu;\beta}^\alpha U^\mu \quad (\text{I.1a})$	$F_G^\beta = -\mathbb{H}_\alpha^\beta S^\alpha; \quad \mathbb{H}_\beta^\alpha \equiv \star R_{\mu\beta\nu}^\alpha U^\mu U^\nu \quad (\text{I.1b})$
Eqs. Magnetic Tidal Tensor	Eqs. Gravito-magnetic Tidal Tensor
$B_\alpha^\alpha = 0 \quad (\text{I.2a})$	$\mathbb{H}_\alpha^\alpha = 0 \quad (\text{I.2b})$
$B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\gamma - 2\pi \epsilon_{\alpha\beta\sigma\gamma} j^\sigma U^\gamma \quad (\text{I.3a})$	$\mathbb{H}_{[\alpha\beta]} = -4\pi \epsilon_{\alpha\beta\sigma\gamma} J^\sigma U^\gamma \quad (\text{I.3b})$
$B_{\alpha\beta} U^\alpha = 0; \quad B_{\alpha\beta} U^\beta = \epsilon_{\alpha\delta}^{\beta\gamma} E_{[\beta\gamma]} U^\delta \quad (\text{I.4a})$	$\mathbb{H}_{\alpha\beta} U^\alpha = \mathbb{H}_{\alpha\beta} U^\beta = 0 \quad (\text{I.4b})$

accelerated frame we have:

$$\epsilon^{\beta\gamma}_{\alpha\delta} E_{\gamma\beta} U^\delta = \epsilon^{\beta\gamma}_{\alpha\delta} E_{\gamma;\beta} U^\delta \Rightarrow \epsilon^{ikj} E_{jk} = (\nabla \times \vec{E})^i. \quad (20)$$

Expressing also the second member of (19a) in terms of the fields E^α and B^α measured in this frame, we obtain, in 3-vector notation:

$$\nabla \times \vec{E} = -\frac{D\vec{B}}{d\tau} - \vec{a} \times \vec{E}, \quad (21)$$

which is a generalization of *Maxwell-Faraday equation* $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$ for accelerated frames, cf. Eq. (20) of [13].

The fact that the gravitoelectric tidal tensor $\mathbb{E}_{\alpha\beta}$ is symmetric again means that there is no analogous gravitational induction effect, and this is a key observation for the applications in Secs. V A and VI.

Eqs. (I.4) are the time projections of the tidal tensors with respect to *the observer* U^α *measuring them* (more precisely, since U^α is the particle's 4-velocity, it is the time projection in the CM frame); they are zero in the gravitational case, as $\mathbb{H}_{\alpha\beta}$ is a spatial tensor, and non-zero in the electromagnetic case, which again is related to electromagnetic induction, as the right Eq. (I.4a) corresponds to the spatially projected Eq. (19a). This means that F_G^α is spatial with respect to U^α , and F_{EM}^α is not, which has important implications on the work done by the fields on the particle, which will be discussed in Sec. V.

Finally, note that $F_{EM}^\alpha = B^{\beta\alpha} \mu_\beta$ is the covariant generalization of the familiar textbook 3-D expression $\vec{F}_{EM} = \nabla(\vec{\mu} \cdot \vec{B})$, the latter being valid only in the *inertial* frame *momentarily* comoving with the particle.

C. Spin “precession” - exact analogy based on inertial GEM fields from the 1+3 formalism

For purely magnetic dipoles, Eq. (16), for the spin evolution under Mathisson-Pirani condition, simplifies to:

$$\frac{D_F S_\mu}{d\tau} = \epsilon_{\mu\alpha\beta\nu} U^\nu \mu^\alpha B^\beta, \quad (22)$$

or, equivalently:

$$\frac{DS_\mu}{d\tau} = S_\nu a^\nu U_\mu + \epsilon_{\mu\alpha\beta\nu} U^\nu \mu^\alpha B^\beta, \quad (23)$$

where $a^\alpha \equiv DU^\alpha/d\tau$ denotes the acceleration, and B^β is the magnetic field *as measured by the test particle*. The first term in (23) embodies the Thomas precession; these equations tell us that, in the absence of electromagnetic field, S^α undergoes Fermi-Walker transport. The second term is a covariant form for the familiar torque $\tau = \vec{\mu} \times \vec{B}$ causing the Larmor precession of a magnetic dipole under the influence of a magnetic field; such precession is generically described by Eq. (22).

Consider now an orthonormal frame $e_{\hat{a}}$ carried by an observer of 4-velocity U^α , such that $U = e_{\hat{0}}$, comoving with the test particle. In such frame, $S^{\hat{0}} = 0$ and $U^{\hat{\alpha}} = \delta_0^{\hat{\alpha}}$, and equation (23) reduces to (see [34]):

$$\frac{DS^{\hat{i}}}{d\tau} = (\vec{\mu} \times \vec{B})^{\hat{i}} \Leftrightarrow \frac{dS^{\hat{i}}}{d\tau} = \left(\vec{S} \times \vec{\Omega} + \vec{\mu} \times \vec{B} \right)^{\hat{i}} \quad (24)$$

where $\vec{\Omega}$ is angular velocity of rotation of the spatial axis $e_{\hat{i}}$ relative to the tetrad Fermi-Walker transported along the center of mass worldline. Note the analogy between the two terms of the second equation. We cast the analogy between the evolution of the spin vector of a gyroscope and a magnetic dipole subject to a magnetic field in three levels; the similarity between the two terms of (24) is the first level. If $B^{\hat{i}} = 0$, what Eq. (24) in fact tells is that \vec{S} does not precess relative to the local spacetime geometry, since $DS^{\hat{i}}/d\tau = 0$. Thus, the so-called “precession” of the gyroscope is an artifact of the reference frame, with no *local* physical meaning; it is simply the spatial rotation of the chosen frame relative to the local spacetime geometry, or the local “compass of inertia” [3] (note that gyroscopes are objects that oppose changes in direction; hence torque-free gyroscopes are supposed to follow the local compass of inertia). That does not mean, however, that it is necessarily meaningless, as it can tell us about non-local *physical* effects. Up until now, the spatial angular velocity of rotation of the tetrad, $\vec{\Omega}$, is arbitrary, since the only condition we imposed was $e_{\hat{0}}$ to

be parallel to the 4-velocity of a given observer. Of special interest for the study of the frame dragging effect is the case where the tetrad field is locked to the so-called “frame of the distant stars”. Such frame, for a stationary, asymptotically flat spacetime, is set up as follows. Consider the congruence of static observers (cf. point 8 of Sec. ID): the *rigid* congruence of observers whose worldlines are tangent to the time-like Killing vector field that, *at infinity*, corresponds to observers at rest in the asymptotic inertial rest frame of the source. These observers are interpreted as being “at rest” with respect to the distant stars; since the congruence is rigid, it may be thought of as a grid of points rigidly fixed with respect to them. This fixes the times axis of the local tetrads of the frame. Now one needs to ensure that each local spatial triad e_i is locked to this grid; that is done by demanding the rotation $\vec{\Omega}$ of the local tetrads (relative to Fermi-Walker transport) equals the vorticity of the congruence (this defines the *frame adapted to the congruence* [54, 55]). In this way the tetrads point towards fixed neighboring observers, see [34] for details; and the local tetrads themselves are locked to one another and to the tetrads adapted to inertial frames at infinity, whose axes define star-fixed directions. Hence $\vec{\Omega}$, which is minus the angular velocity of precession of a gyroscope relative to such frame, yields minus its precession rate relative to the distant stars, or to the inertial frame at infinity. It is this construction that allows one to mathematically define the “precession” of a gyroscope relative to the distant stars (i.e., to determine the rotation, relative to the local guiding gyroscope, of the frame whose axis are locked to the distant stars), which was the effect measured by the Gravity Probe B mission [56]. This is an effect which is physical, clearly distinguishing, e.g., the Kerr from the Schwarzschild spacetimes, even though it has no local significance and can only be measured by locking to the distant stars, by means of telescopes (notice that the choice of tetrad above is precisely equivalent to requiring that static observers see the distant stars fixed in their skies).

When $\vec{\Omega}$ equals the vorticity of the congruence, then $\vec{\Omega} = \vec{H}/2$, where \vec{H} is the “gravitomagnetic field”; this designation is due³ to the fact that it plays in the *exact* geodesic equations (see [34]) the same role as the mag-

netic field \vec{B} in the electromagnetic Lorentz force. This is the second level of the analogy in Eq. (24). The third level of the analogy can be read from the field equations for \vec{H} , which exhibit striking similarities with Maxwell equations for \vec{B} in an accelerated, rotating frame; for a detailed account of this analogy we refer the reader to the companion paper [34]. A well known manifestation is the correspondence between the Lense-Thirring effect and electromagnetism: the gravitomagnetic field produced by a spinning mass, as measured by the congruence of static observers, is similar to the magnetic field produced by a spinning charge.

If we assume $\vec{\mu} = \sigma \vec{S}$, then the quantity $S^2 = S^\alpha S_\alpha = S^{\alpha\beta} S_{\alpha\beta}/2$ is a constant of the motion, which is immediately seen contracting (23) with S^μ . This result might be puzzling, as it apparently contradicts our assessments in Secs. II E and V, where we conclude that $dm/d\tau$ is a variation of kinetic energy of rotation, via electromagnetic induction. If the kinetic energy of rotation varies, S must also vary. However, as explained in Sec. VI, such contradiction is but an artifact of the pole-dipole approximation, which takes into account the work done by the induced electric field on the test particle (which involves its magnetic dipole moment $\vec{\mu}$), but neglects the torque it exerts (which depends on the particle’s second moment of the charge, that is of quadrupole order). This also explains why the analogy manifest in Eq. (24) is valid for arbitrary fields, while most gravito-electromagnetic analogies based on GEM vector fields (not tidal tensors!), such as the geodesic equation/Lorentz force, e.g. [5, 9], or the forces on gyroscopes/magnetic dipoles [5, 34], break down when one considers time-dependent fields [34, 57] (another exception is the hidden momentum analogy, presented in the next section).

D. Momentum of the spinning particle - hidden momentum and exact analogy based on inertial GEM fields from the 1+3 formalism

The momentum (4) of a spinning particle is not in general parallel to its center of mass 4-velocity U^α ; one may obtain an expression for P^α contracting (9) with U^β , yielding

$$P^\alpha = mU^\alpha - \frac{DS^{\alpha\beta}}{d\tau} U_\beta + \mu^{\mu\alpha} E_\mu, \quad (25)$$

where $m \equiv -U^\alpha P_\alpha$ and $E^\alpha = F^\alpha_\beta U^\beta$ is the electric field *as measured by the test particle* of 4-velocity U^α . We split P^α in its projections parallel and orthogonal to the CM 4-velocity U^α :

$$P^\alpha = P^\alpha_{\text{kin}} + P^\alpha_{\text{hid}}; \quad P^\alpha_{\text{kin}} \equiv mU^\alpha, \quad P^\alpha_{\text{hid}} \equiv (h^U)^\alpha_\beta P^\beta. \quad (26)$$

We dub the parallel projection $P^\alpha_{\text{kin}} = mU^\alpha$ “kinetic momentum” associated with the motion of the center of mass. This is the most familiar part of P^α , formally similar to the momentum of a monopole particle. The

³ In the framework of the GEM fields from the 1+3 formalism, the force on the gyroscope [5, 34] and the equation for the geodesic motion of a (non-spinning) test particle (see e.g. [5, 9]) can be exactly described by equations analogous to the ones from electromagnetism. These results are not however as general as Eq. (24): in the case of the force on the gyroscope, the analogy holds only for stationary fields, and if the gyroscope is at rest with respect to some stationary observer (i.e., if the *gyroscope’s worldline* is tangent to a time-like Killing vector); in the case of geodesics, the analogy holds only (again) for stationary fields and stationary observers (i.e., the *observer’s worldline* must be tangent to a time-like Killing vector). This is discussed in detail in [34].

component P_{hid}^α orthogonal to U^α is the so-called “hidden momentum” [29]. The reason for the latter denomination is seen taking the perspective of an observer $\mathcal{O}(U)$ comoving with the particle: in the frame of $\mathcal{O}(U)$ (i.e., the frame $U^i = 0$) the 3-momentum is in general not zero: $\vec{P} = \vec{P}_{\text{hid}} \neq 0$; however, by definition, the particle’s CM is at rest in that frame; hence this momentum must be somehow hidden in the spinning particle.

With the Mathisson-Pirani condition $S^{\alpha\beta}U_\beta = 0$, the momentum P^α in Eq. (25) becomes:

$$P_{\text{kin}}^\alpha = mU^\alpha; \quad P_{\text{hid}}^\alpha = -\epsilon_{\beta\gamma\delta} S^\beta a^\gamma U^\delta + \epsilon_{\beta\gamma\delta} \mu^\beta E^\gamma U^\delta. \quad (27)$$

It is useful to introduce the following notation for the two parts of P_{hid}^α :

$$P_{\text{hidEM}}^\alpha \equiv \epsilon_{\beta\gamma\delta} \mu^\beta E^\gamma U^\delta; \quad P_{\text{hidI}}^\alpha \equiv -\epsilon_{\beta\gamma\delta} S^\beta a^\gamma U^\delta \quad (28)$$

which we shall now explain. Eq. (27) tells us that $P_{\text{hid}}^\alpha \neq 0$ if the particle is accelerating and/or subject to an electric field. In the particle’s CM frame (where $U^i = 0$), and in vector notation, its space part reads ($P_{\text{hid}}^0 = 0$):

$$\vec{P}_{\text{hid}} = \vec{P} = -\vec{S} \times \vec{a} + \vec{\mu} \times \vec{E} = \vec{S} \times \vec{G} + \vec{\mu} \times \vec{E}, \quad (29)$$

where $\vec{G} = -\vec{a}$ is the gravito-electric field as defined in [5, 9, 34]. The term $\vec{P}_{\text{hidEM}} = \vec{\mu} \times \vec{E}$ is a still not well known feature of relativistic electrodynamics (despite its discovery by Shockley & James [58] dating back from the 60’s, and having since been discussed in number of papers, e.g. [58–61], including recent ones [29, 62]). It is a form of hidden momentum that is originated by the action of the electromagnetic field; but it is however *purely mechanical in nature*, as explained in Appendix D using a simple model. \vec{P}_{hidEM} equals *minus* the electromagnetic momentum \vec{P}_\times generated by the dipole when placed in an external electromagnetic field, which, in the particle’s frame, reads (see e.g. [59, 63] and Appendix D):

$$\vec{P}_\times = \int \vec{E} \times \vec{B}_{\text{dipole}} = -\vec{\mu} \times \vec{E} = -\vec{P}_{\text{hidEM}}.$$

This means that in flat space-time, for example, the spatial momentum of a dipole whose center of mass is *at rest* is in general not zero in the presence of an electromagnetic field; it is actually \vec{P}_{hidEM} which allows for the total spatial momentum $\vec{P}_{\text{tot}} \equiv \vec{P}_{\text{matter}} + \vec{P}_{\text{EM}}$ to vanish in this stationary configuration, as required by the conservation equations $(T_{\text{tot}})^{\alpha\beta}_{;\beta} = 0$.

\vec{G} is a field of “inertial forces”; hence the term $\vec{P}_{\text{hidI}} = \vec{S} \times \vec{G}$ is the “inertial” analogue of $\vec{\mu} \times \vec{E}$ (for this reason hereafter we will dub it “inertial hidden momentum”), and the formal similarity between the two can be understood through an analogue model, as discussed in detail in Appendix D. However, an important difference between these two forms of hidden momentum must be noted: whereas P_{hidEM}^α is a physical, gauge invariant

quantity, P_{hidI}^α is pure gauge, and depends only on the frame where the center of mass is computed. That is what we are going to show next.

Consider a generic spin condition $S^{\alpha\beta}u_\beta = 0$, where u^α denotes the 4-velocity of an arbitrary observer $\mathcal{O}(u)$. As discussed in Sec. II A, this condition means that we take as reference worldline the center of mass as measured by $\mathcal{O}(u)$. For simplicity, let us consider $F^{\alpha\beta} = 0$ (so that $P_{\text{hid}}^\alpha = P_{\text{hidI}}^\alpha$); in this case, contracting (9) with u_β , and using $S^{\alpha\beta}u_\beta = 0$, leads to [25]

$$S^{\alpha\beta} \frac{Du_\beta}{d\tau} = \gamma(U, u) P^\alpha - m(u) U^\alpha \quad (30)$$

where $\gamma(U, u) \equiv -U^\alpha u_\alpha$ and $m(u) \equiv -P^\alpha u_\alpha$. Hence, if $Du^\alpha/d\tau \neq 0$, then in general the momentum is not parallel to the 4-velocity: $P^{[\alpha}U^{\beta]} \neq 0$, and thus the particle will have *hidden momentum*. This comes as a natural consequence of what we discussed in Sec. II A (see also [25] for further details): the position of the CM of a spinning body depends on the vector u^α relative to which it is computed; if that vector varies along the reference worldline it is clear that this is reflected in the velocity U^α of the CM, which in general will accelerate even without the action of any forces. Also U^α will in general no longer be parallel to P^α , and thus the centroid is not at rest in the frame $P^i = 0$. But if we take a field u^α such that $Du^\alpha/d\tau = 0$, i.e., if we take as reference worldline the center of mass as measured with respect to a field u^α that *is parallel transported along it*, then $P^\alpha = mU^\alpha = P_{\text{kin}}^\alpha$, and there is no hidden momentum.

Thus we see that indeed P_{hidI}^α is pure gauge (it is worth noting that the condition $P^\alpha \parallel U^\alpha$ is actually cast in [40] as one of the possible spin supplementary conditions). This also means that the motion effects induced by it (such as the bobbings studied in [29]) must be confined to the worldtube of centroids (i.e. the worldtube formed by all the possible positions of the center of mass, as measured by every possible observer; for flat spacetime, it is a tube of radius S/M centered around the center of mass measured in the zero 3-momentum frame, see [25, 40] for details), so that they can be made to vanish by a suitable choice of reference worldline. For this reason it is dubbed in [29], where it was first discussed, “kinematical hidden momentum” (by contrast with gauge invariant hidden momentum P_{hidEM}^α , dubbed “dynamic” therein). In flat spacetime, if one measures the center of mass with respect to an inertial frame (since u^α is in this case covariantly constant), then $P_{\text{hidI}}^\alpha = 0$.

The inertial hidden momentum also plays a key role in the helical solutions allowed by the condition $S^{\alpha\beta}U_\beta = 0$, as discussed in detail in the companion paper [25]; namely it explains how the motion of a spinning particle can be consistently described by an helical solution without violating any conservation principle. And using the analogy in Eq. (29), we show in [25] that these solutions are a phenomena which can be cast as analogous to the bobbing of a magnetic dipole in an external electric field, studied in Sec. III.B.1 of [29].

E. Mass of the spinning particle

We take the scalar $m = -P^\alpha U_\alpha$ as “the proper mass”⁴ [49] of the spinning particle. It is simply the time projection of P^α in the particle’s proper frame, i.e., the particle’s energy as measured in the frame where it has zero 3-velocity. Whereas for a non-spinning particle m is a constant of the motion, for a spinning particle, that is not true in general, as we shall see. It follows from the definition of m that:

$$\frac{dm}{d\tau} = -\frac{DP^\alpha}{d\tau}U_\alpha - P^\alpha a_\alpha = -\frac{D_F P^\alpha}{d\tau}U_\alpha \quad (31)$$

i.e., $dm/d\tau$ is the time projection, in the CM frame, of the Fermi-Walker derivative of the momentum. Noting that $P^\alpha a_\alpha = P_{\text{hidEM}}^\alpha a_\alpha$, and using the orthogonality $P_{\text{hidEM}}^\alpha U_\alpha = 0$, we can rewrite this equation as:

$$\frac{dm}{d\tau} = -\left(\frac{DP^\alpha}{d\tau} - \frac{DP_{\text{hidEM}}^\alpha}{d\tau}\right)U_\alpha \quad (32)$$

Thus $dm/d\tau$ equals also the time projection, in the CM frame, of the force $DP^\alpha/d\tau$ *subtracted* by the derivative of the “electromagnetic” hidden momentum $DP_{\text{hidEM}}^\alpha/d\tau$. Let us see the meaning of the first term. Contracting (17) with U^α we obtain:

$$\frac{DP^\alpha}{d\tau}U_\alpha = B^{\beta\alpha}\mu_\beta U_\alpha = \frac{D\star F^{\beta\gamma}}{d\tau}U_\gamma\mu_\beta \quad (33)$$

showing that the force has a time projection if the Maxwell tensor varies along the CM worldline. That arises from the force on a magnetic dipole, Eq. (I.1a) of Table I (by contrast with its gravitational counterpart (I.1b), which is spatial with respect to U^α). As we shall see in Sec. V A, from the point of view of the inertial frame momentarily comoving with the particle, this time-projection has the interpretation of the rate of work done by the electric field induced by the varying magnetic field. Hence, in the absence of an electromagnetic field, $U_\alpha DP^\alpha/d\tau = P^\alpha a_\alpha = 0$, and therefore m is conserved. If an electromagnetic field is present, in general m varies. Now, noting from Eqs. (27) and (2) that $P^\alpha a_\alpha = \star F^{\beta\gamma}a_\gamma\mu_\beta$, and putting Eqs. (31) and (33) together, we see that:

$$\frac{dm}{d\tau} = -\frac{D}{d\tau}(\star F^{\beta\gamma}U_\gamma)\mu_\beta \equiv -\mu_\mu \frac{DB^\mu}{d\tau} \quad (34)$$

⁴ This is the most natural definition of the body’s mass if one uses the Pirani spin condition, since it is the quantity which is conserved in the absence of electromagnetic field, as we shall see next. However, it should be noted that if one uses Dixon’s condition $S^{\alpha\beta}P_\beta = 0$ instead, then it is the quantity $M \equiv \sqrt{-P^\alpha P_\alpha}$ (not m), i.e., the energy of the particle as measured in the zero 3-momentum frame, that is conserved when $F^{\alpha\beta} = 0$ (see Appendix C).

If we take $\mu^\alpha = \sigma S^\alpha$, being σ a constant, and, since from Eq. (23), $B^\mu DS_\mu/d\tau = 0$, we have [19, 20, 39, 46]:

$$\frac{dm}{d\tau} = -\sigma \frac{D}{d\tau}(S_\mu B^\mu) \quad (35)$$

$$\Rightarrow m = m_0 - \sigma S_\mu B^\mu = m_0 - \sigma \vec{S} \cdot \vec{B} \quad (36)$$

where m_0 is a constant. Thus, if $\vec{\mu} = \sigma \vec{S}$, m is the sum of a constant plus the quantity $-\vec{\mu} \cdot \vec{B}$ which is commonly dubbed in elementary textbooks as the “magnetic potential energy” of the dipole; for this reason some authors (e.g. [22], [46] p. 18, [36] p. 1622) have interpreted this term as meaning that the magnetic potential energy contributes to the mass. We argue (in agreement with the analysis in [64–67]), that what it actually means, in rigor, is that *there is no such thing as magnetic potential energy*. $-\vec{\mu} \cdot \vec{B}$ is *internal energy* of the test particle; in fact it will be shown in Sec. VI A 2 (cf. also [65–67]), that, for a “rigid” body, it is but rotational kinetic energy, associated with the rotation of the body around its center of mass. What the term $-\vec{\mu} \cdot \vec{B}$ actually does is to ensure that *the work done by the magnetic field on a magnetic dipole is zero* (hence no potential energy can be assigned to it), as is discussed in detail in Sec. V.

Note that this solves an apparent paradox that has for long been discussed in the literature [39, 64, 65, 67]: consider a magnetic dipole in a static non-homogeneous magnetic field \vec{B} ; a net force $\vec{F}_{\text{EM}} = \nabla(\vec{\mu} \cdot \vec{B})$ will be exerted on it, therefore it gains kinetic energy; but, on the other hand, we know that the magnetic field can do no work, because if we think about the dipole as a current loop (cf. Fig. 3a below) and consider the force exerted in each of its individual moving charges, we see that the magnetic force $\vec{F} = q(\vec{v} \times \vec{B})$ is always orthogonal to the velocity \vec{v} of the charges, so that no work can be done. Consider herein, for simplicity (the general case will be discussed in detail in Sec. V B), a setup where $P_{\text{hid}}^\alpha = 0$, and a globally inertial frame in flat spacetime; in that case, the variation of the particle’s kinetic energy of translation is *exactly* canceled out by a variation of the particle’s proper mass m , so that the particle’s energy $E = -P_0$ (as measured in a frame where the field is time-independent⁵) is *constant*. If we take, as discussed above, a “rigid” spinning body as our dipole model, we can say that the variation of translational kinetic energy, associated to the body’s CM motion, is exactly canceled out by a variation of kinetic energy of rotation (about the body’s CM). This is in agreement with the arguments in [64–67], which are re-formulated and generalized, in a relativistic covariant framework, in Secs. V B and VI A

⁵ If in the reference frame chosen the magnetic field is time-dependent, the energy $E = -P_0$ of the particle, as measured in that frame, varies in general, but not through the work of the magnetic field; it is due to the work of the induced electric field, see Sec. V and e.g. [64, 68].

below (see in particular Sec. VI C). In order to make these important points more clear, it is useful to compare with the cases of a monopole charged particle, and an electric dipole subject to an electromagnetic field, which is done in Appendix B 4. There is also a force on the particle, which is set into motion gaining kinetic energy; but, in these cases, *the electric field is doing work*, there is a potential energy involved, and the gain in kinetic energy of the center of mass is *not* canceled out by a variation of the particle's proper mass (m is constant for a monopole particle, and also for an electric dipole if one assumes that the dipole vector is parallel transported⁶).

Another point we would like to emphasize is that the varying mass m , and the contribution $-\vec{\mu} \cdot \vec{B}$ to it, are something real and physically measurable, not just a matter of semantics or definition of the particle's mass (i.e. not an issue that goes away by simply calling instead m_0 , in Eq. (36) above, the particle's mass): m is the inertial mass of the particle. In order to see that, take, again for simplicity, the case that $P_{\text{hid}}^\alpha = 0$; we have

$$\frac{DP^\alpha}{d\tau} = ma^\alpha + \frac{dm}{d\tau} U^\alpha ;$$

i.e., the projection of the force, orthogonal to U^α , is ma^α (thus, in the CM frame, $D\vec{P}/d\tau = m\vec{a}$). This inertial mass is measurable, for instance in collisions. Also (since, as mentioned above, in the case of a quasi-rigid spinning body, $-\vec{\mu} \cdot \vec{B}$ is kinetic energy of rotation) the angular velocity of rotation of a spinning body is measurable.

In the purely gravitational case, by contrast, the proper mass is a constant ($m = m_0$); this means, conversely, that, unlike its electromagnetic counterpart, the gravito-magnetic field *does work* on mass currents. This point will be further explored and exemplified in Sec V B.

F. Center of mass motion

Eqs. (I.1) yield the *force* on the spinning particle in the electromagnetic and gravitational case; *not* the acceleration $a^\alpha \equiv DU^\alpha/d\tau$, as $P^\alpha \neq m_0 U^\alpha$. Putting $m \equiv m_0 + m'$ in Eq. (27), and noting that:

$$\epsilon_{\beta\gamma\sigma}^\alpha \mu^\beta E^\gamma U^\sigma = \star F^{\beta\alpha} \mu_\beta + \mu^\beta B_\beta U^\alpha \quad (37)$$

where we have used the decomposition (2), we can write P^α as:

$$P^\alpha = m_0 U^\alpha - \epsilon_{\beta\gamma\delta}^\alpha S^\beta a^\gamma U^\delta + (m' + \mu^\alpha B_\alpha) U^\alpha + \star F_\beta^\alpha \mu^\beta$$

This is simplified if we assume $\mu^\alpha = \sigma S^\alpha$; in that case, as described in the previous section, $m' = -\mu^\alpha B_\alpha$ and thus:

$$P^\alpha = m_0 U^\alpha - \epsilon_{\beta\gamma\delta}^\alpha S^\beta a^\gamma U^\delta + \star F_\beta^\alpha \mu^\beta$$

Therefore:

$$m_0 a^\alpha = \frac{DP^\alpha}{d\tau} + \epsilon_{\beta\gamma\delta}^\alpha \frac{D}{d\tau} U^\delta (S^\beta a^\gamma) + \star F_{\beta;\tau}^\alpha U^\tau \mu^\beta - \star F_\beta^\alpha \frac{D\mu^\beta}{d\tau} \quad (38)$$

In a region where the charge current density j^α is zero (most of the applications in this paper deal with vacuum) we have, according to Eq. (I.3), $\star F_{\alpha\beta;\tau} U^\tau = 2B_{[\alpha\beta]}$. Therefore, if $j^\alpha = 0$, and using (17):

$$m_0 a^\alpha = q F^{\alpha\beta} U_\beta + B^{\alpha\beta} \mu_\beta - \mathbb{H}^{\beta\alpha} S_\beta - \star F_\beta^\alpha \frac{D\mu^\beta}{d\tau} + \epsilon_{\beta\gamma\delta}^\alpha U^\delta \frac{D}{d\tau} (S^\beta a^\gamma) \quad (39)$$

Note the reversed indices in the second term, comparing to the expression for the force (I.1a). This leads to a counter-intuitive dynamical behavior, as exemplified in Sec. IV A.

III. WEAK FIELD REGIME: GRAVITATIONAL SPIN-SPIN FORCE

If the fields do not vary along the test particle's world-line (so that $\star F_{\alpha\beta;\gamma} U^\gamma = 0$), Eqs. (I.3) allow for a similarity between F_{EM}^α and F_G^α , since $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ have the same symmetries. Consider the simple example of analogous physical apparatus: a magnetic dipole in the electromagnetic field of a spinning charge (charge Q , magnetic moment μ_s), and a gyroscope in the gravitational field of a spinning mass (mass M , angular momentum J), asymptotically described by the Kerr solution. In this application we shall be interested in the far field limit — where the non-linearities of the gravitational field are negligible, and one may expect a similarity between the gravitational and electromagnetic interactions — and in the slow motion regime.

We start by briefly describing the approximations we will use. We take as gravitational field the usual linearized Kerr metric, Eq. (13) of [6], with the gravitational “potentials” $\Phi \equiv M/r$, $\vec{\mathcal{A}} \equiv \vec{J} \times \vec{r}/r^3$, and $\Theta_{ij} \equiv \Phi \delta_{ij}$ (herein for simplicity we will be using nearly Lorentz coordinates, so that \hat{g}_{ij} in [6] becomes δ_{ij}). The electromagnetic potentials by their turn are *exactly* $\phi \equiv Q/r$ and $\vec{A} \equiv \vec{\mu}_s \times \vec{r}/r^3$. The gravitational tidal tensors, are, consistently, linearized in the “fields”; in other words, these tensors consist of i) derivatives of the metric potentials, ii) products of the latter by the metric potentials, and iii) products between the derivatives; the terms of the type ii) and iii) are dropped, only the type i) are kept. The

⁶ In a more general case, the proper mass of an electric dipole is not constant, but its variation substantially differs from its magnetic counterpart (which can be traced back to the intrinsic difference between the two dipole models, being a magnetic dipole a current loop, and the electric dipole a system of two close monopoles of opposite charges); it does not depend on the time derivative of the electric field seen by the test particle, and does not cancel the work done by the electric field. The case of electric dipoles is discussed in detail in Appendix B.

electromagnetic tidal tensors are linear in the potentials, hence no weak field assumption is made in the forces (40), (42)-(43). The expression for the acceleration (46), however, involves a term of second order in the fields, which is to be neglected in order to make a coherent comparison with linearized gravity. In the computation of the electromagnetic and gravitational tidal tensors involved in the forces (42)-(43), (44)-(45), exerted on slowly moving test particles (velocity v), only terms up to first order in v are kept (as usual in slow motion approximations). Terms of second order in S^2 , involved in accelerations (46), are neglected, as usual practice in Post Newtonian and weak field treatments, e.g. [29, 69] (often without a clear justification), as they are of the same order of magnitude as the quadrupole contributions, as shown below.

The general expressions for the magnetic $B_{\alpha\beta}$ and the linearized gravitomagnetic $\mathbb{H}_{\alpha\beta}$ tidal tensors are given by Eqs. (12) and (17)-(20) of [6]; the expressions corresponding to the problem at hand are obtained by substituting therein the potentials above. Let us first consider stationary setups; i.e. the test particle to be at rest relative to the static observers (may be thought as the observers at rest with respect to the central source; see point 8 of Sec. ID); that amounts to set the 4-velocity $u^\alpha \simeq \delta_0^\alpha$ in Eqs. (12), (17)-(20) of [6]. For these observers, the gravitational tidal tensors asymptotically match the electromagnetic ones, identifying the appropriate parameters [57]:

$$(E^u)_{ij} \simeq \frac{M}{r^3} \delta_{ij} - \frac{3Mr_i r_j}{r^5} \stackrel{M \leftrightarrow Q}{=} (E^u)_{ij},$$

$$(\mathbb{H}^u)_{ij} \simeq 3 \left[\frac{(\vec{r} \cdot \vec{J})}{r^5} \delta_{ij} + 2 \frac{r_{(i} J_{j)}}{r^5} - 5 \frac{(\vec{r} \cdot \vec{S}) r_i r_j}{r^7} \right]$$

$$\stackrel{J \leftrightarrow \mu_s}{=} (B^u)_{ij}$$

(all the time components vanish identically for these observers). Therefore, for large r , the force exerted on a gyroscope *whose center of mass is at rest* relative to the central mass is similar to its electromagnetic counterpart, identifying $\mu_s \leftrightarrow J$ and $\mu \leftrightarrow S$:

$$F_G^i = -\mathbb{H}^{ji} S_j \stackrel{\{J,S\} \leftrightarrow \{\mu_s, \mu\}}{\simeq} -F_{\text{EM}}^i \quad (40)$$

Thus, for stationary setups, we have a gravitational spin-spin force similar to its electromagnetic counterpart, which was first found by Wald in [1].

Manifestation of the different symmetries — in the general case that the dipole/gyroscope are allowed to move, however, Eqs. (I.3) make clear that the two forces differ significantly, due to the different symmetries of the tidal tensors. These differences (which lead to key differences in the dynamics, as exemplified in next sections) *are manifest even in the weak field and slow motion approximation*, as we shall now show. Consider now the test particles to be moving with 3-velocity \vec{v} relative to

the central sources⁷. The magnetic tidal tensor as seen by the moving dipole, $B_{\alpha\beta}$, can be obtained in terms of the tidal tensors $(E^u)_{\alpha\beta}$, $(B^u)_{\alpha\beta}$ measured by the static observers, using the decomposition

$$\star F_{\alpha\beta;\gamma} = 2U_{[\alpha} B_{\beta]\gamma} - \epsilon_{\alpha\beta\mu\sigma} U^\sigma E^\mu_\gamma. \quad (41)$$

The force (I.2a) exerted on it reads, *to first order* in v

$$F_{\text{EM}}^i \simeq B^{ji} \mu_j \simeq (B^u)^{ji} \mu_j - (E^u)^{li} \epsilon^j_{kl} v^k \mu_j, \quad (42)$$

$$F_{\text{EM}}^0 = B^{i0} \mu_i = 0. \quad (43)$$

The gravitomagnetic tidal tensor as seen by the moving gyroscope, $\mathbb{H}_{\alpha\beta}$, can analogously be obtained in terms of the tidal tensors $(\mathbb{E}^u)_{\alpha\beta}$, $(\mathbb{H}^u)_{\alpha\beta}$ measured by the static observers, using the decomposition (the dual of (74) below):

$$\star R_{\alpha\beta}{}^{\gamma\delta} = 4\epsilon^{\lambda\tau}_{\alpha\beta} u_\lambda u^{[\gamma} (\mathbb{E}^u)_{\tau}{}^{\delta]} - 2\epsilon^{\tau}_{\alpha\beta} [{}^{[\gamma} (\mathbb{E}^u)_{\tau}{}^{\delta]} + 4(\mathbb{H}^u)_{[\beta}{}^{[\delta} u^{\gamma]} u_{\alpha]} + \epsilon^{\lambda\tau}_{\alpha\beta} \epsilon^{\gamma\delta\mu\nu} (\mathbb{H}^u)_{\mu\tau} u_\lambda u_\nu$$

The force exerted on the gyroscope reads, *to linear order* in the fields, and also to first order in v :

$$F_G^i \simeq -\mathbb{H}^{ji} S_j \simeq -(\mathbb{H}^u)^{ji} S_j + 2(\mathbb{E}^u)^{li} \epsilon^j_{kl} v^k S_j, \quad (44)$$

$$F_G^0 \simeq -\mathbb{H}^{i0} S_i \simeq -(\mathbb{H}^u)^{ji} v_j S_i. \quad (45)$$

These expressions can also be obtained by setting (together with the above substitutions for the electromagnetic and gravitational potentials) $u^\alpha \rightarrow U^\alpha \simeq (1, \vec{v})$ in the expressions for the corresponding tidal tensors Eqs. (12), (17)-(20) of [6]. For this application however the expressions in the form given above will suit us most.

We note that, to this accuracy, the spatial part of the forces, apart from global signs and a factor of two in the second term of (44) compared to (42), differ essentially in the fact that the former is symmetrized, whereas the latter is not. The dynamical consequences of this fact are exemplified in Sec. IV below. Also the time components are different: $F_{\text{EM}}^0 = -(F_{\text{EM}})_0 = 0$ (this is an exact result), telling us that the total work done on the dipole, as measured by the static observers, is zero; but $(F_G)_0 \neq 0$, telling us that there is a non-vanishing (and non-negligible in the linear regime) work being done on the gyroscope. Also one may check that whereas $F_G^\alpha U_\alpha = 0$ (i.e., F_G^α is a spatial force), F_{EM}^α has a non-vanishing time projection in the CM frame which, to first order in v , reads: $F_{\text{EM}}^\alpha U_\alpha \simeq (B^u)^{ji} \mu_j v_i$. The physical significance of the time components of the forces is discussed in Sec. V.

⁷ Since the gravitational field we are dealing with is the linearized Kerr solution, which does not embody a central spinning mass, on more general grounds one would describe the gyroscope motion by saying it moves with velocity \vec{v} relative to the static observers, i.e., observers of 4-velocity $u^\alpha = u^0(1, 0)$, tangent to $\partial/\partial t$, in Boyer Lindquist coordinates.

It should also be noted that the forces above do not translate in a trivial fashion into accelerations, cf. Sec. IIF; F^α in general is not even parallel to a^α , as the test particles possess hidden momentum. Assuming the proportionality $\mu^\alpha = \sigma S^\alpha$, from Eq. (39) we have in the electromagnetic case

$$\begin{aligned} m_0 a^i &\simeq F_{\text{EM}}^i + 2B^{[ij]}\mu_j \simeq B^{ij}\mu_j \\ &\simeq (B^u)^{ij}\mu_j - (E^u)^{lj}\epsilon^i{}_{kl}v^k\mu_j. \end{aligned} \quad (46)$$

The last two terms of (39) are herein neglected. As for the term $\star F_\beta^i D\mu^\beta/d\tau$, for $\mu^\alpha = \sigma S^\alpha$, it follows from Eq. (23) that it is of second order in the fields; since in the gravitational case we are dealing with linearized theory, we must coherently consider electromagnetic fields weak enough so that only their linear contributions are relevant to the dynamics. The term $DP_{\text{hidI}}^\alpha/d\tau = \epsilon^\alpha{}_{\beta\gamma\delta}U^\delta D(S^\beta a^\gamma)/d\tau$ is also negligible in this approximation if, among the many possible solutions allowed by the condition $S^\alpha\beta U_\beta = 0$ (see e.g. [25]), we choose the “non-helical” one; actually imposing $P_{\text{hidI}}^\alpha \approx 0$ amounts, *in this application* (not in general⁸), to ensure that we pick such solution. This is justified as follows.

For a non-helical representation, the acceleration comes in first approximation from the force F_{EM}^α subtracted by $DP_{\text{hidEM}}^\alpha/d\tau$; i.e., from the terms we kept in (46). Since $|(B^u)^{ij}\mu_j| \sim \sigma S\mu_s/r^4$, and $|(E^u)^{lj}\epsilon^i{}_{kl}v^k\mu_j| \sim \sigma SQv/r^3$, we have

$$\left| \frac{D(\vec{S} \times_U \vec{a})}{d\tau} \right| \sim \frac{\sigma S^2 v \mu_s}{m r^5} + \frac{\sigma S^2 v^2 Q}{m r^4} \sim v \frac{R}{r} |(B^u)^{ij}\mu_j|$$

where we used: $D(\vec{S} \times_U \vec{a})/d\tau \approx \vec{S} \times_U D\vec{a}/d\tau$, since $\vec{a} \times_U D\vec{S}/d\tau$ involves terms quadratic in the fields by virtue of Eqs. (23) and (46); $dr/dt = \vec{v} \cdot \vec{r}/r$; $S \sim mv_{\text{rot}}R$, and we allow v_{rot} (the rotation velocity of a point at the body’s surface) to take its maximum value $v_{\text{rot}} = 1$; R is the radius of the spinning test body. The second term of the second expression above is neglected since it is quadratic in v . The order of magnitude of the contribution $D\vec{P}_{\text{hidI}}/d\tau$ to the acceleration (46), corresponds thus to a multiplication of the magnitude of first term of Eq. (46) by a small factor R/r , being $r \gg R$ ensured by the far field assumptions. It is worth noting also that, independently of any weak field or slow motion consideration, if no assumptions are made about the multipole structure of the body, the contribution P_{hidI}^α to P^α could not even be kept in a consistent pole-dipole approximation, as it is of the typical order of magnitude of the quadrupole contributions to the hidden momentum, which are to be neglected; this is discussed in detail in Appendix C 2.

In the gravitational system we have, from Eq. (39):

$$m_0 a^i \simeq F_G^i \simeq -(\mathbb{H}^u)^{ji}S_j + 2(\mathbb{E}^u)^{li}\epsilon^j{}_{kl}v^kS_j \quad (47)$$

Again the last term of (39) is negligible if we assume the non-helical representation, as by arguments entirely analogous to the ones above we show it is of the order $\sim |(\mathbb{H}^u)^{ji}S_j|R/r$. But note that in the purely gravitational case (by contrast with electromagnetism in general, see Footnote 8), we can actually impose (to this degree of accuracy) the condition $P_{\text{hidI}} \approx 0$ as a means of picking the non-helical representation. Note also that (since there is no other type of hidden momentum) this effectively amounts to impose the momentum velocity relation $P^\alpha \simeq mU^\alpha$.

We shall now compare these results with the ones recently presented in [29]. We note that Eq. (46) agrees with Eq. (111) of [29], making $q = 0$, neglecting quadrupole terms in the latter, and identifying $B^{i;j} = (B^u)^{ij}$ and $E^{i;j} = (B^u)^{ij}$, being \vec{E} and \vec{B} , in the notation of [29], the electric and magnetic fields measured by the static observers. It then reads in our notation:

$$ma^i = (B^u)^i{}_j\mu^j - \epsilon_{jkl}v^k(E^u)^{li}\mu^j + \epsilon^i{}_{lj}(E^u)^l{}_k v^k\mu^j \quad (48)$$

And Eq. (47) agrees with Eq. (113) of [29], provided that the quadrupole terms and terms quadratic in v in the latter are neglected, and one identifies $\mathcal{B}^{i;j} = \mathbb{H}^{ij}/2$, $\mathcal{E}^{i;j} = \mathbb{E}^{ij}$, where $\vec{\mathcal{E}}$ is the gravitoelectric field ($\vec{\mathcal{E}} = -\vec{G}$ in the notation of [34]), $\vec{\mathcal{B}}$ is the gravitomagnetic field ($\vec{\mathcal{B}} = -\vec{H}/4$ in the notation of [34]), it reads in our notation

$$ma^i = -(\mathbb{H}^u)^j{}_i S^j + 2\epsilon_{jkl}v^k(E_G)^{li}S^j - \epsilon^i{}_{lj}(E_G)^l{}_k v^k S^j \quad (49)$$

The pair of Eqs. (48)-(49), is equivalent to (46)-(47), only written in a different form. The main difference is that the former makes linearized gravity look very similar to electromagnetism; whereas the latter makes more transparent key differences between the two interactions. Namely the crucial ones which occur when the test particle moves (even in the slow motion, weak field regime) and are manifest in the fact that the tensor in the second term of (47) is symmetrized, by contrast with (46). In the form (48)-(49) of [29], by their turn, for the problem at hand, they formally differ only by a factor of 2 second term of (49) (the term $\partial^i[\vec{S} \cdot (\vec{v} \times \vec{E}_G)]$ in the notation of [29]), comparing with the corresponding term in (48). This factor alone accounts for the aforementioned differences in this formulation, which tends to mask them. The examples in Secs. IV A and IV exemplify how dramatic they can be.

Thus, to conclude: in the literature concerning the weak field gravito-electromagnetic analogy (e.g. [2]) the gravitational force acting on a gyroscope is commonly cast in the form $\vec{F}_G = \nabla(\vec{H} \cdot \vec{S})/2$ (being \vec{H} the “gravitomagnetic field”, defined in those treatments as $H^i \equiv \epsilon^i{}_{jk}g_{0k,j}$), similar to its electromagnetic counter-

⁸ In general one cannot a priori neglect the term $\vec{S} \times_U \vec{a}$ based on the arguments above; for instance if the particle is charged and suffers a Lorentz force, cf. first term, of Eq. (39); that yields a contribution to the acceleration of zeroth order in S .

part $\vec{F}_{\text{EM}} = \nabla(\vec{B} \cdot \vec{\mu})$ (the latter being an exact expression, holding in the *inertial* frame momentarily comoving with the CM, where \vec{B} and $\vec{\mu}$ are measured, c.f. Sec. II B), seemingly implying a similarity between the two interactions. We emphasize that such expressions are not suited to describe the dynamics of a gyroscope, as they can hold only if the gyroscope's center of mass is at *rest* in a *stationary* field. This is usually overlooked in the literature, despite in [1], where this analogy was originally presented, being mentioned that it was derived precisely under those conditions. Otherwise (if the fields are time dependent and/or the test particle moves) the gravitational tidal tensors $\mathbb{H}_{\alpha\beta}$ involved are very different from their electromagnetic counterparts $B_{\alpha\beta}$, even in the slow motion regime, as one can see with full generality substituting the expressions for tidal tensors (12) and (17) – (20) of [6] in Eqs (I.1). Even if one analyzes the system from the test particle's frame (which is where its electromagnetic counterpart exactly holds), it is clear that $F_G^i = -\mathbb{H}^{ji}S_j \neq \nabla^i(\vec{H} \cdot \vec{S})/2$ (it contains more terms), see Eq. (20) of [6].

In this application we have seen that the two forces indeed are similar when the test particles are at rest with respect to the sources, which allowed us to cast gravitational the force for this case as a spin-spin force, in analogy with electromagnetism. However in the general dynamical case the test particle is allowed to move, the two forces differ significantly even to first order in the velocity (and in the fields), cf. Eqs. (48)-(49). That is, the “spin-orbit” interaction of gravity and electromagnetism are very different.

Sec. III in brief

- In the *stationary*, weak field regime, when the particles are at rest with respect to the sources, the interactions are very similar, and there is a gravitational spin-spin force analogous to the electromagnetic one.
- When the particles move, differences imposed by the symmetries of the tidal tensors are important and non-negligible in any slow motion approximation.

IV. DYNAMICAL MANIFESTATIONS OF THE SYMMETRIES OF THE MAGNETIC TIDAL TENSORS

A. Radial motion in Schwarzschild spacetime

According to Eqs. (I.1) of Table I, it is the magnetic tidal tensor, *as seen by the test particle* of 4-velocity $U^\alpha = U^0(1, \vec{v})$, that determines the force exerted upon it. Consider first a magnetic dipole in the field of a static

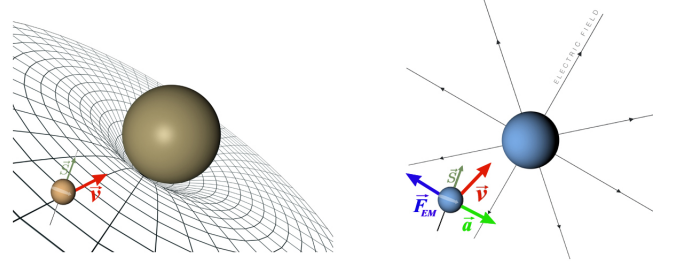


Figure 1: An illustration of the physical consequences of the different symmetries of the tidal tensors. A gyroscope dropped from rest in Schwarzschild spacetime will move radially along a geodesic towards the source, with no force exerted on it. A magnetic dipole in (initially) radial motion in a Coulomb field, by contrast, suffers a force. Due to the hidden momentum, the force is approximately *opposite* to the acceleration!

point charge Q , and with a purely radial initial velocity. Unlike what one might naively expect, a force will be exerted on the dipole; the radially moving dipole indeed sees a vanishing magnetic field B^α , but a nonvanishing magnetic tidal tensor $B_{\alpha\beta}$, which can be seen from the expressions (57) below. Taking the perspective of the frame comoving with the particle, this is explained through the laws of electromagnetic induction: the moving dipole “sees” a time-varying electric field; by virtue of Eq. (I.3a) (which in vacuum is a covariant form for $\nabla \times \vec{B} = \partial \vec{E} / \partial t$), that will induce a curl in \vec{B} , i.e., an antisymmetric part in the magnetic tidal tensor $B_{\alpha\beta}$. Which implies that the tensor $B_{\alpha\beta}$ is itself non-vanishing:

$$B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\gamma \neq 0 \Rightarrow B_{\alpha\beta} \neq 0$$

and therefore (except for the special case $\vec{v} \parallel \vec{\mu}$) a net force will be exerted on the dipole; computing it explicitly we obtain:

$$F_{\text{EM}}^0 = 0; \quad F_{\text{EM}}^i = B^{[ai]} \mu_\alpha = \frac{\gamma Q}{r^3} (\vec{v} \times \vec{\mu})^i. \quad (50)$$

Note this important point: for this configuration, actually $B_{\alpha\beta} = B_{[\alpha\beta]}$, i.e., the force F_{EM}^α comes *entirely* from the antisymmetric part of $B_{\alpha\beta}$. Let us now consider the analogous gravitational setup, a gyroscope *in radial motion* in Schwarzschild spacetime. Since the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$ is symmetric (in vacuum): $\mathbb{H}_{[\alpha\beta]} = 0$, we expect, in the spirit of the analogy, $\mathbb{H}_{\alpha\beta}$ as “seen” by the gyroscope to vanish under these conditions, and therefore, by virtue of Eq. (I.1b) of Table I:

$$F_G^\alpha = -\mathbb{H}^{\beta\alpha} S_\beta = 0$$

Indeed this is the case, as one may check computing explicitly $\mathbb{H}^{\beta\alpha} S_\beta = \star R^{\beta\mu\alpha\nu} U_\mu U_\nu S_\beta$, with $U^\alpha = (U^0, U^r, 0, 0)$, for this metric. This means that a gyroscope in radial motion moves along a geodesic (it is the trivial solution of the equations of motion with Pirani

condition, see Appendix C 1), with no forces applied on it. For instance, a gyroscope dropped from rest will fall into the singularity moving in a straight line.

Finally, note that the force in the electromagnetic problem, due the hidden momentum, does not translate in a simple fashion into an acceleration. This is manifest in Eq. (39); for flat spacetime, and a particle whose only electromagnetic moment is μ^α ($q = 0$), it reads:

$$m_0 a^\alpha = B^{\alpha\beta} \mu_\beta + \epsilon^\alpha_{\beta\gamma\delta} \frac{D}{d\tau} (S^\beta a^\gamma) U^\delta - \star F_\beta{}^\alpha \frac{D\mu^\beta}{d\tau}.$$

The last term vanishes if one assumes $\vec{\mu} = \sigma \vec{S}$, since: $B^\alpha(U) = 0$ for radial motion; thus, from Eq. (23), $D\mu_\mu/d\tau = S_\nu a^\nu U_\mu$ and $\star F_\beta{}^\alpha D\mu^\beta/d\tau = -B(U)^\alpha S_\nu a^\nu = 0$. The second term can also be taken to a good approximation as being zero, which, as explained in Sec. III, to an accuracy of order $\mathcal{O}(S^2)$, amounts to say that we pick the “non-helical” solution allowed by the Mathisson-Pirani condition. Therefore, since, in this application, $B_{(\alpha\beta)} = 0$, we are led to the conclusion that $m_0 a^\alpha \approx -F_{\text{EM}}^\alpha$!

B. Equatorial motion in Kerr and Kerr-de-Sitter spacetimes

In this application we compare equatorial motions of gyroscopes in the Kerr and Kerr-de-Sitter spacetimes to magnetic dipoles in the field of a spinning charge, to point out another important physical consequence of the different symmetries of the tidal tensors: whereas in the electromagnetic case $B_{\alpha\beta}$ can never vanish for a moving dipole, as a consequence of the laws of electromagnetic induction, in the gravitational case there is a velocity field for which $\mathbb{H}_{\alpha\beta} = 0$, i.e., a gyroscopes moving with such velocities feel no force. This opens the possibility of circular geodesics for spinning particles, which we show to exist in Kerr-de-Sitter spacetimes. In order to obtain those results we make use of another exact gravito-electromagnetic analogy [11–13] (other than the tidal tensor and the inertial force analogies of Secs. IIB and IIC), this time a purely *formal* one (cf. [14]), between the decomposition of the Maxwell tensor $F^{\alpha\beta}$ in electric and magnetic fields and the decomposition of the Riemann tensor (in vacuum) in electric and magnetic tidal tensors. In particular, from the formal analogy between the scalar invariants of $F^{\alpha\beta}$ and the second order invariants of $R_{\alpha\beta\gamma\delta}$ (which will be studied in more detail in the forthcoming paper [15]) we note that in the equatorial plane of Kerr spacetime the gravito-magnetic *tidal tensor* vanishes for some velocity field, in analogy with the vanishing of the magnetic *field* (not the magnetic tidal tensor!) in the corresponding electromagnetic system. The later means that a magnetic dipole with those velocities does not precess (apart from Thomas precession); for completeness we also investigate the analogous gravitational problem — check if there is a velocity for gyroscopes such that they do not “precess” relative to

the distant stars; the answer is affirmative, and turns out to asymptotically match its electromagnetic counterpart. This result will be of importance in the context of the physical significance of curvature the invariants, and the clarification of their relationship with gravitomagnetism, to be discussed in [15].

1. A magnetic dipole in the field of a spinning charge.

Velocity field for which $B^\alpha = 0$. — We start by the electromagnetic system which will serve as a guide for the gravitational case. With respect to a unit timelike 4-vector u^α , the Maxwell tensor (6 independent components) splits irreducibly into the two spatial vectors (3 independent components each) $(E^u)^\alpha \equiv F^{\alpha\beta} u_\beta$ and $(B^u)^\alpha \equiv \star F^{\alpha\beta} u_\beta$, cf. decomposition (1), which are a co-variant definition for, respectively, the electric and magnetic fields as measured by an observer of 4-velocity u^α . Despite both $(E^u)^\alpha$ and $(B^u)^\alpha$ depending on the observer 4-velocity u^α , combining them one can construct the two second order scalar invariants (e.g. [11, 12, 70]):

$$E^\alpha E_\alpha - B^\alpha B_\alpha = -\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \quad (51)$$

$$E^\alpha B_\alpha = -\frac{1}{4} F_{\alpha\beta} \star F^{\alpha\beta} \quad (52)$$

These are actually the only two independent scalar invariants one can construct from the Maxwell tensor. They have the following physical interpretation [15, 70, 71]: i) if $E^\alpha B_\alpha \neq 0$ then the electric E^α and magnetic B^α fields are both non-vanishing for all observers; ii) if $E^\alpha E_\alpha - B^\alpha B_\alpha > 0$ (< 0) and $E^\alpha B_\alpha = 0$, there are observers u^α for which the magnetic field $(B^u)^\alpha$ (the electric field $(E^u)^\alpha$) is zero.

The electromagnetic field produced by a spinning charge (magnetic moment $\vec{\mu}_s$) is described by the 4-potential $A^\alpha = (\phi, \vec{A})$:

$$\phi = \frac{Q}{r}, \quad \vec{A} = \frac{1}{c} \frac{\vec{\mu}_s \times \vec{r}}{r^3} = \frac{1}{c} \frac{\mu_s \sin \theta}{r^3} \vec{e}_\phi. \quad (53)$$

The invariant structure for this electromagnetic field is:

$$\begin{cases} \vec{E}^2 - \vec{B}^2 = \frac{Q^2}{r^4} - \frac{\mu_s^2 (5 + 3 \cos 2\theta)}{2r^6} > 0, \\ \vec{E} \cdot \vec{B} = \frac{2\mu_s Q \cos \theta}{r^5} (= 0 \text{ in the equatorial plane}) \end{cases} \quad (54)$$

which tells us that in the equatorial plane $\theta = \pi/2$ there are observers who measure B^α to be *locally* zero (since $\vec{E} \cdot \vec{B} = 0$ and $\vec{E}^2 - \vec{B}^2 > 0$ therein). It is straightforward to obtain the 4-velocity of such observers. The magnetic field $(B^u)^\alpha \equiv \star F^{\alpha\beta} u_\beta$ seen by an arbitrary observer of

4-velocity $u^\alpha = (u^t, u^r, u^\theta, u^\phi)$ is given by:

$$B^r = \frac{2\mu_s \cos \theta}{r^3} u^t, \quad B^\theta = \left(\frac{\mu_s u^t}{r^4} - \frac{u^\phi Q}{r^2} \right) \sin \theta, \\ B^\phi = \frac{Q u^\theta}{r^2 \sin \theta}, \quad B^t = \frac{\mu_s}{r^3} (2u^r \cos \theta + r u^\theta \sin \theta).$$

Thus, the condition $B^r = 0$ implies $\theta = \pi/2$ (i.e., equatorial plane, as expected); in the equatorial plane, $B^t = 0$ implies $u^\theta = 0$, and $B^\theta = 0$ implies $u^\phi/u^t = \mu_s/(Qr^2)$. If we additionally assume that the charge and mass are identically distributed in the body, its gyromagnetic ratio is $\mu_s/J = Q/2M$, and we therefore conclude that observers with angular velocity

$$v^\phi = \frac{u^\phi}{u^t} = \frac{J}{2Mr^2} \equiv v_{(\mathbf{B}=0)}^\phi, \quad (55)$$

see a vanishing magnetic field in the equatorial plane. No restriction is imposed on the radial component of the velocity, apart from the normalization condition $u^\alpha u_\alpha = -1$; that is, the velocity for which the magnetic field vanishes at a given point is not unique—it comprehends a class of observers [57]. The velocity field corresponding to the case $u^r = 0$ is plotted in Fig. 2a). The vanishing of B^α for these observers comes from an exact cancellation between the magnetic field generated by the relative translation of the source and the one produced by its rotation. And it means that a magnetic dipole possessing a velocity of the form (55) does not undergo Larmor precession, since the second term of Eq. (23) vanishes.

Hence dipoles moving in circular lines tangent to the velocity field of Eq. (55) undergo Fermi-Walker transport. Note that this does not mean they do not “precess” relative to the distant stars (in this case, a global inertial frame); they do, since such worldlines are accelerated, due to Thomas precession. The latter, from the point of view of the inertial frame, is encoded in the first term of Eq. (23); and from the point of view of the frame comoving with the dipole, where Eq. $DS^{\hat{i}}/d\tau = 0$ holds, and with axes fixed to the distant stars, it is encoded in the connection $\Gamma_{\hat{0}\hat{j}}^{\hat{i}} = \epsilon_{\hat{i}\hat{k}\hat{j}} \Omega^{\hat{k}}$, cf. Eq. (24). Let us denote hereafter by $\vec{\Omega}_\star$ the angular velocity of rotation, relative to Fermi-Walker transport along the particle’s worldline, of a comoving *spatial triad fixed to the distant stars*; and by $-\vec{\Omega}_{\star\text{tot}}$ the total precession rate of the particle’s spin relative to this frame. Thus, cf. Eq. (24):

$$\frac{d\vec{S}}{d\tau} = \vec{\Omega}_{\star\text{tot}} \times \vec{S}; \quad \vec{\Omega}_{\star\text{tot}} = -\vec{\Omega}_\star - \sigma \vec{B} \quad (56)$$

In flat spacetime, $\vec{\Omega}_\star$ equals *minus* the Thomas precession; it can be shown (e.g. [63] p. 564) that $\vec{\Omega}_\star = \gamma \vec{v} \times \vec{a}/(\gamma^2 + 1)$, where \vec{v} is the velocity of the dipole relative to the distant stars, and \vec{a} the spatial part of the acceleration $a^\alpha = DU^\alpha/d\tau$. Hence, $\Omega_\star = 0$ *only* for a magnetic dipole in *straightline motion*, momentarily *moving with a velocity obeying* (55). This observation

is important for the comparison with the gravitational analogue in Sec. IV B 2.

$B_{\alpha\beta}$ *never vanishes*.— The force (I.1a) exerted on the dipole, however, does *not* vanish, as it is only the field, *not its derivatives* (or its tidal tensor), that vanishes. Actually, the magnetic tidal tensor $B_{\alpha\beta}$, *as measured by a moving dipole, never vanishes*. As seen by a generic observer, it has following components in the equatorial plane:

$$B_{r\theta} = \alpha(r^2 Q u^\phi - 3\mu_s u^t); \quad B_{\theta r} = \alpha(2r^2 Q u^\phi - 3\mu_s u^t); \\ B_{r\phi} = -\alpha Q r^2 u^\theta; \quad B_{\phi r} = -2\alpha Q r^2 u^\theta; \\ B_{\theta\phi} = \alpha Q r^2 u^r; \quad B_{\phi\theta} = -\alpha Q r^2 u^r \\ B_{tr} = 3\alpha \mu_s u^\theta; \quad B_{t\theta} = 3\alpha \mu_s u^r \quad (57)$$

with $\alpha \equiv 1/r^3$. Thus we see that in order to make $B_{(\alpha\beta)}$ vanish, we must have $u^\theta = u^r = 0$ and

$$v^\phi = \frac{u^\phi}{u^t} = 2 \frac{\mu_s}{Q r^2} = \frac{J}{M r^2} \equiv v_{(B_{(\alpha\beta)}=0)}^\phi \quad (58)$$

(differs from a factor of 2 from the velocity (55) which makes B^α vanish; the second equality again assumes that the charge and mass are identically distributed in the source). However, $B_{[\alpha\beta]}$ only vanishes if $\vec{v} = 0$; hence it is not possible to find an observer for which $B_{\alpha\beta} = B_{(\alpha\beta)} + B_{[\alpha\beta]} = 0$. Again, the fact that $B_{[\alpha\beta]}$ cannot vanish for a moving observer is a direct consequence of Maxwell’s equations/the laws of electromagnetic induction: a dipole moving relative to the spinning charge always sees a varying electromagnetic field; that endows $B_{\alpha\beta}$ with an antisymmetric part, by virtue (from the point of view of the MCRF) of vacuum equation $\nabla \times \vec{B} = \partial E/\partial t$, or covariantly, by Eq. (I.3a). Note that this is true even if one considers a dipole in circular equatorial trajectories around the central source: $D \star F_{\alpha\beta}/d\tau = 2B_{[\alpha\beta]} \neq 0$ along such worldline, which is due to the variation of the electric field along the curve (it is constant in magnitude, but varying in direction).

2. A gyroscope in Kerr spacetime

Velocity field for which the gyroscope does not “precess”. — Drawing a parallelism with the previous section, we start by posing the question: is there a gravitational analogue to the velocity field (55) (i.e., a cancellation between translational and rotational gravitomagnetism)?

Before answering this question one must first note some subtleties. What we did in the previous section was finding a velocity field, given by (55), for which $B^\alpha = 0$; so to study the analogous gravitational problem, the first difference to note is that herein the problem does not translate into the vanishing of physical field such as the magnetic field B^α . As explained in detail in the companion paper [34], the gravitomagnetic field \vec{H} , which comes at first glance as the most natural candidate if one is to

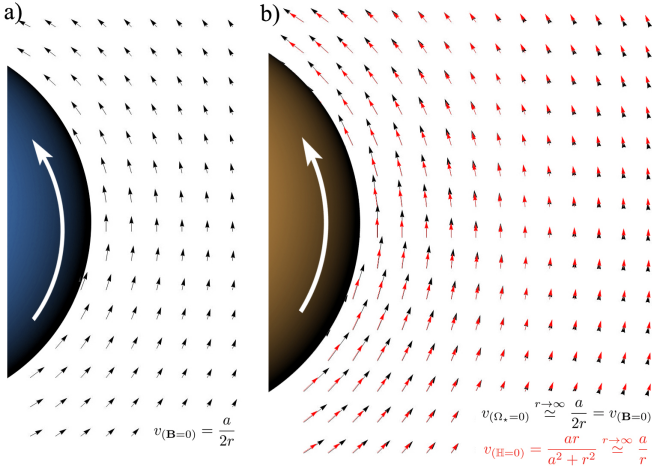


Figure 2: a) Equatorial plane of a spinning charge. Black arrows: velocity field $\vec{v}_{(\mathbf{B}=0)}$, which makes the *magnetic field* B^α vanish; magnetic dipoles in straightline motion, *momentarily* with such velocities, do not precess relative to the distant stars. b) Equatorial plane of Kerr spacetime. Black arrows: velocity field $\vec{v}_{(\Omega_*=0)}$ for which a gyroscope in straightline motion momentarily does not “precess” (with respect to the distant stars); asymptotically it matches its electromagnetic counterpart. Red arrows: velocity field $\vec{v}_{(\mathbb{H}=0)}$ which makes the *gravito-magnetic tidal tensor* $\mathbb{H}_{\alpha\beta}$ vanish; means that gyroscopes moving along trajectories tangent to $\vec{v}_{(\mathbb{H}=0)}$ feel no force, $F_G^\alpha = 0$. If $\Lambda > 0$ (Kerr-dS spacetime) circular geodesics for gyroscopes do even exist (Sec. IV B 3). $\vec{v}_{(\mathbb{H}=0)}$ has no electromagnetic analogue: for a moving dipole *always* $B_{\alpha\beta} \neq 0$, by virtue of Eq. $B_{[\alpha\beta]} = \star D F_{\alpha\beta} / d\tau$, generically implying $F_{\text{EM}}^\alpha \neq 0$.

draw a parallelism with electromagnetism, *is not* a physical field, it is a mere artifact of the reference frame; so obviously it is not reflected in any field invariants, and therefore an analysis in the spirit of the one in the previous section is immediately ruled out (an analogous invariant analysis can be used instead to solve for the vanishing of the gravitomagnetic *tidal tensor* $\mathbb{H}_{\alpha\beta}$, as we shall see below). Moreover, unlike B^α , which depends only on a given observer’s 4-velocity (not on its derivatives), \vec{H} is defined with respect to a congruence of observers, and consists on the sum of the vorticity $\vec{\omega}$ of the observer congruence plus the rotation $\vec{\Omega}$ of the local tetrad relative to Fermi-Walker transport [34]. Thus in the most general case \vec{H} can be anything, depending on the reference frame one wishes to set up. Hence, more than some naive notion of gravitomagnetic field, what makes sense here to compare the are the *physical effects*. We know that a gyroscope at finite r in Kerr spacetime “precesses” with respect to a frame anchored to the distant stars (how such a frame is set up is explained in Secs. II C and in [10, 34]). This reflects an effect — frame dragging — which is physical (even though it has no local meaning, and can only be measured by locking to the distant stars by means of a telescope), and analogous to the precession of a magnetic dipole subject to a magnetic field, as

discussed in Sec. II C. The only part involved in it is $\vec{\Omega}$; which (in this case that the reference tetrad is locked to the distant stars) gains thereby a physical meaning, that we can compare with electromagnetism.

The total precession of the test particle relative to the distant stars (56) is thus a quantity one can define both for the magnetic dipole of the previous section and for the gyroscope in Kerr spacetime. Hence it will serve as a guide to find the gravitational analogue of the velocity field (55). Firstly we note that, as discussed in the previous section, the velocity field (55) means *not* that $\vec{\Omega}_{\star\text{tot}}$ vanishes for a dipole moving in circular lines tangent to $\vec{v}_{(\mathbf{B}=0)}$ ($\vec{\Omega}_\star \neq 0$ in that case due to the Thomas precession); but that $\vec{\Omega}_{\star\text{tot}} = 0$ for a dipole carried by an observer moving in a *straight line* with a velocity that *momentarily* coincides with $\vec{v}_{(\mathbf{B}=0)}$. So the analogous problem in gravity seems to be finding the velocities at which observers in straightline motion carrying spatial triads *with axes fixed to the distant stars* must travel, in order for such triads to be Fermi-Walker transported (i.e., $\vec{\Omega}_\star = \vec{\Omega}_{\star\text{tot}} = 0$, so that a gyroscope it carries does not “precess” relative to the distant stars). However a difference must be noted: in the electromagnetic case the vanishing of $\vec{\Omega}_{\star\text{tot}}$ comes from the fact that by moving in straight-line, the Thomas precession vanishes, and by having a velocity *momentary* obeying (55) the Larmor precession $-\sigma\vec{B}$ also vanishes; the two appearing in Eq. (24) as clearly separate terms. In the gravitational case, by contrast, we only have $\vec{\Omega}_\star = -\vec{\Omega}_{\star\text{tot}}$, which is itself the *total* rate of rotation of the reference tetrad relative to Fermi-Walker transport, and *embodies* the Thomas precession. The latter is not zero for straightline motion in curved spacetime⁹, since these are accelerated observers in general. Hence one might wonder, which one is the problem analogous to the vanishing of B^α for the velocity field (55) of the previous section: the velocity field such that $\vec{\Omega}_\star = 0$ for a gyroscope moving with zero acceleration, or moving in “straight line”? (The result will depend on this choice, due to the Thomas precession, that contributes in the latter case). We argue it is the latter, since, as will be shown below, it is the one that asymptotically matches the electromagnetic counterpart, and coincides with vanishing of the Post-Newtonian (PN) gravitomagnetic field \vec{H}_{PN} , as defined in the comoving PN frame (note that \vec{H}_{PN} is meaningful¹⁰ in this context, since the PN frames are anchored to

⁹ In fact it is not clear how one should define “straight line” in a curved spacetime, since light rays and spacelike geodesics are similarly deflected by the gravitational field. In the case of the equatorial plane for the Kerr solution, however, axisymmetry provides a good definition of “radial direction” and “tangential direction”, which we can explore to define straight line, see the discussion following Eqs. (64).

¹⁰ One might wonder whether the *exact* problem at hand could be solved using the gravitomagnetic field \vec{H} , i.e., instead of basing

the distant stars; note also that the Thomas precession is cast as a part of \vec{H}_{PN} [72]). It can thus be interpreted as a cancellation between *translational* and *rotational* gravitomagnetism, just like the vanishing of B^α for observers moving with velocities $\vec{v}_{(\mathbf{B}=0)}$ arises from an exact cancellation between the magnetic fields generated by the rotation and relative translational motion of the source.

We shall now compute such velocity field. The Boyer-Lindquist line element for $\theta = \pi/2$ can be written as

$$ds^2 = -V \left(dt + \frac{2Ma}{rV} d\phi \right)^2 + W^{-1} dr^2 + r^2 W V^{-1} d\phi^2$$

where $V = 1 - \frac{2M}{r}$ and $W = \frac{\Delta}{r^2} = 1 - \frac{2M}{r} + \frac{a^2}{r^2}$. This form of the metric suggests the orthonormal coframe

$$\begin{aligned} \omega^{\hat{0}} &= V^{\frac{1}{2}} dt + \frac{2Ma}{rV^{\frac{1}{2}}} d\phi; \\ \omega^{\hat{r}} &= W^{-\frac{1}{2}} dr; \quad \omega^{\hat{\phi}} = rW^{\frac{1}{2}} V^{-\frac{1}{2}} d\phi, \end{aligned} \quad (59)$$

whose dual frame $\{E_{\hat{0}}, E_{\hat{r}}, E_{\hat{\phi}}\}$ can be defined geometrically. Indeed, $E_{\hat{0}}$ is simply the unit vector tangent to $\partial/\partial t$, which is the the unique (up to a constant) Killing vector field which is timelike at infinity. Thus the congruence with four-velocity $E_{\hat{0}}$ corresponds to the only observers who can claim to be “at rest” with respect to the distant stars — the static observers, cf. point 8 of Sec. I D. The vectors $E_{\hat{r}}$ and $E_{\hat{\phi}}$ can be geometrically defined by the condition of being respectively orthogonal and tangent to the surfaces of constant r . These surfaces have geometric meaning, as they are the orbits of the equatorial plane’s isometry group (generated by the Killing vector fields $\partial/\partial t$ and $\partial/\partial \phi$).

the analysis in the local tetrad associated to the single observer carrying the gyroscope, one could embed it (in some neighborhood) in a congruence, and then take the frame adapted to it ($\vec{\omega} = \vec{\Omega}$), in order to define a gravitomagnetic field $\vec{H} = 2\vec{\omega}$. That would seemingly allow us to take advantage of the analogy between the transformation laws for the gravitomagnetic and magnetic fields (see Eqs. (4.14) and (8.3) of [9]), in the spirit of what is done in [72]; and to solve this problem by simply demanding $\vec{H}(U) = 0$, just like the in the electromagnetic case the solution (55) amounts to solve for \vec{v} the Eq. $\vec{B}(U) = 0$. This scheme however is of no use in our exact problem, since the \vec{H} computed by this procedure does not tell us about the precession relative to the distant stars; what it measures is the rotation of a tetrad locked to the vorticity of the congruence relative to local Fermi-Walker transport. This is not in general a comparison of the latter with the “distant star” fixed tetrad (only in very special cases, see Sec. (II C) and [34]), and in particular it is never so if the congruence has shear. One may check that the vorticity (thus the \vec{H} of the adapted frame) of the congruence of “boosted” observers (66) does not match the rotation $\vec{\Omega}_*$ (relative to Fermi-Walker transport), of the boosted tetrads $\vec{E}_{\hat{\alpha}}$ fixed to the distant stars: $\vec{\omega} = \vec{H}/2 \neq \vec{\Omega}_*$. The scheme would only work in the weak field (Post Newtonian) regime, and for PN boosted frames (frames with constant *coordinate* velocity), since to PN accuracy these frames are fixed to the distant stars, and their shear is negligible (only the expansion is not), see [34].

To compute the Fermi-Walker transport law we need the connection coefficients $\Gamma_{\hat{\mu}\hat{\nu}}^{\hat{\alpha}}$, defined, as usual, by $\nabla_{E_{\hat{\mu}}} E_{\hat{\nu}} = \Gamma_{\hat{\mu}\hat{\nu}}^{\hat{\alpha}} E_{\hat{\alpha}}$. They can be codified in the *Cartan connection forms*

$$\omega_{\hat{\beta}}^{\hat{\alpha}} = \Gamma_{\hat{\gamma}\hat{\beta}}^{\hat{\alpha}} \omega^{\hat{\gamma}}, \quad (60)$$

which form the unique solution to the *Cartan structure equations*

$$\omega_{\hat{\alpha}\hat{\beta}} = -\omega_{\hat{\beta}\hat{\alpha}}; \quad d\omega^{\hat{\alpha}} + \omega_{\hat{\beta}}^{\hat{\alpha}} \wedge \omega^{\hat{\beta}} = 0. \quad (61)$$

A straightforward calculation yields

$$\omega_{\hat{r}}^{\hat{0}} = \omega_{\hat{0}}^{\hat{r}} = \frac{M}{r^2} V^{-1} W^{\frac{1}{2}} \omega^{\hat{0}} - \frac{Ma}{r^3} V^{-1} \omega^{\hat{\phi}}; \quad (62)$$

$$\omega_{\hat{\phi}}^{\hat{0}} = \omega_{\hat{0}}^{\hat{\phi}} = \frac{Ma}{r^3} V^{-1} \omega^{\hat{r}};$$

$$\omega_{\hat{r}}^{\hat{\phi}} = -\omega_{\hat{\phi}}^{\hat{r}} = \frac{Ma}{r^3} V^{-1} \omega^{\hat{0}} +$$

$$\frac{1}{r} \left(1 - \frac{4M}{r} + \frac{4M^2}{r^2} - \frac{Ma^2}{r^3} \right) V^{-1} W^{-\frac{1}{2}} \omega^{\hat{\phi}}. \quad (63)$$

We now have to decide how to define the concept of “pointing in the same direction” for spacelike vectors orthogonal to $E_{\hat{0}}$. Clearly $E_{\hat{r}}$ and $E_{\hat{\phi}}$ do not point in the same direction throughout the equatorial plane, because they fail to do so even in the flat Minkowski case $M = a = 0$. By analogy with what one would do in this case, we consider the frame

$$\begin{aligned} E_0 &= E_{\hat{0}}; \\ E_1 &= \cos \phi E_{\hat{r}} - \sin \phi E_{\hat{\phi}}; \\ E_2 &= \sin \phi E_{\hat{r}} + \cos \phi E_{\hat{\phi}}, \end{aligned} \quad (64)$$

This frame, which we will call the *fiducial frame*, will define what we mean by “pointing in the same direction with respect to the distant stars”. We will say that a gyroscope does not precess with respect to the distant stars if the components of its spin vector in the fiducial frame remain constant¹¹. Moreover, this frame will also define what is meant by “straight line motion”. We will say that a particle is moving in straight line with constant velocity if the components of its velocity vector in the fiducial frame remain constant. From the Lie brackets

$$[E_0, E_{\hat{r}}] = \frac{M}{r^2} V^{-1} W^{\frac{1}{2}} E_0; \quad [E_0, E_{\hat{\phi}}] = 0,$$

we see that $[E_0, E_1]$ and $[E_0, E_2]$ are proportional to E_0 , meaning that the distribution generated by E_0 and any linear combination of E_1 and E_2 with constant coefficients is integrable. In other words, particles moving

¹¹ Setting up this frame experimentally is another matter; it does not suffice to point E_1 , say, towards the apparent position of some distant star, because of gravitational lensing.

in straight line in the same direction but with different speeds meet the same static observers throughout their paths (albeit at different times). This shows that our definition of straight line motion is consistent.

To compute the connection coefficients of the fiducial frame we notice that its dual coframe is obtained from Eqs. (64) above by replacing $E_{\hat{\alpha}} \rightarrow \omega^{\hat{\alpha}}$, and hence for $\phi = 0$ (which can always be assumed due to the axial symmetry) we have $d\omega^{\hat{0}} = d\omega^{\hat{0}}$;

$$d\omega^{\hat{1}} = d\omega^{\hat{r}} - d\phi \wedge \omega^{\hat{\phi}}; \quad d\omega^{\hat{2}} = d\omega^{\hat{\phi}} + d\phi \wedge \omega^{\hat{r}}.$$

From Cartan's structure Eqs. (61) we obtain

$$\begin{aligned} \omega^{\hat{0}}_{\hat{1}} &= \omega^{\hat{1}}_{\hat{0}} = \omega^{\hat{0}}_{\hat{r}}; & \omega^{\hat{0}}_{\hat{2}} &= \omega^{\hat{2}}_{\hat{0}} = \omega^{\hat{0}}_{\hat{\phi}}; \\ \omega^{\hat{2}}_{\hat{1}} &= -\omega^{\hat{1}}_{\hat{2}} = \omega^{\hat{\phi}}_{\hat{r}} - d\phi = \omega^{\hat{\phi}}_{\hat{r}} - \frac{1}{r}W^{-\frac{1}{2}}V^{\frac{1}{2}}\omega^{\hat{\phi}}. \end{aligned} \quad (65)$$

Consider now the frame obtained from the fiducial frame (64) by a global boost of velocity v in the $E_{\hat{2}}$ direction,

$$\begin{aligned} \tilde{E}_{\hat{0}} &= \gamma E_{\hat{0}} + v\gamma E_{\hat{2}}; \\ \tilde{E}_{\hat{1}} &= \gamma E_{\hat{1}}; \\ \tilde{E}_{\hat{2}} &= v\gamma E_{\hat{0}} + \gamma E_{\hat{2}} \end{aligned} \quad (66)$$

(where $\gamma = 1/\sqrt{1-v^2}$). Its dual coframe is $\tilde{\omega}^{\hat{1}} = \omega^{\hat{1}}$;

$$\tilde{\omega}^{\hat{0}} = \gamma\omega^{\hat{0}} - v\gamma\omega^{\hat{2}}; \quad \tilde{\omega}^{\hat{2}} = -v\gamma\omega^{\hat{0}} + \gamma\omega^{\hat{2}}.$$

It is well known that the matrices Ω and $\tilde{\Omega}$ of the connection forms in the fiducial and boosted frames are related by

$$\tilde{\Omega} = \mathcal{S}^{-1}\Omega\mathcal{S} + \mathcal{S}^{-1}d\mathcal{S}, \quad (67)$$

where

$$\mathcal{S} = \begin{pmatrix} \gamma & 0 & v\gamma \\ 0 & 1 & 0 \\ v\gamma & 0 & \gamma \end{pmatrix} \quad (68)$$

is the change of basis matrix (hence $d\mathcal{S} = 0$ in this case). Therefore we have

$$\tilde{\omega}^{\hat{1}}_{\hat{2}} = v\gamma\omega^{\hat{0}}_{\hat{1}} + \gamma\omega^{\hat{1}}_{\hat{2}}. \quad (69)$$

The components of the spin vector of a gyroscope at rest in the boosted frame will remain constant if and only if $\tilde{E}_{\hat{1}}$ and $\tilde{E}_{\hat{2}}$ are Fermi-Walker transported along $\tilde{E}_{\hat{0}}$, that is, if and only if the space components of $\nabla_{\tilde{E}_{\hat{0}}}\tilde{E}_{\hat{1}}$ and $\nabla_{\tilde{E}_{\hat{0}}}\tilde{E}_{\hat{2}}$ vanish. This is equivalent to having $\tilde{\Gamma}^1_{02} = 0$, that is, to $\tilde{\omega}^{\hat{1}}_{\hat{2}}$ having no component in $\tilde{\omega}^{\hat{0}}$. Using (62), (65) and (69) we can write this condition as

$$\begin{aligned} v^2 + \frac{r^2}{Ma} \left[\left(1 - \frac{5M}{r} + \frac{6M^2}{r^2} - \frac{2Ma^2}{r^3} \right) - V^{\frac{3}{2}} \right] W^{-\frac{1}{2}}v \\ + 1 = 0 \end{aligned} \quad (70)$$

(assuming $a \neq 0$; if $a = 0$, i.e. in the Schwarzschild case, this condition is equivalent to $v = 0$). This equation is quadratic in v , and in principle has two solutions; however, since the product of these solutions is 1, only one of them will correspond to the velocity of the for which $\Omega_{\star} = 0$. In the far field region, where $M/r \ll 1$, (70) approaches

$$v^2 - \frac{2r}{a}v + 1 = 0, \quad (71)$$

whose physically acceptable solution approaches

$$v_{(\Omega_{\star}=0)} \xrightarrow{r \rightarrow \infty} \frac{a}{2r}. \quad (72)$$

Note the striking similarity with the velocity field (55) that makes B^{α} vanish in the analogous electromagnetic problem: both velocities depend only on r and on the ratio $a \equiv J/M$ (if the charge and mass are identically distributed, in the electromagnetic case), and *asymptotically they match*. This weak field limit is the one one would obtain using a boosted Post-Newtonian metric (e.g. [73]), and solving for $H^i_{\text{PN}} = 0$, using the (most common) *naive* notion of gravito-magnetic field defined as the curl of the off-diagonal part of the metric: $\tilde{H}_{\text{PN}} \simeq -\nabla \times \tilde{A}$, $A_i = g_{0i}$, since for an orthonormal tetrad (attached to the background inertial frame, which in this case means the distant stars), the expression $dH^i_{\text{PN}}/d\tau = (\tilde{S} \times \tilde{H}_{\text{PN}})^i/2$ always holds, as shown in [57].

Close to the ergosphere $r = 2M$, on the other hand, (70) approaches

$$v^2 - 2v + 1 = 0 \Leftrightarrow v_{(\Omega_{\star}=0)} = 1, \quad (73)$$

and hence the velocity for which there is no precession approaches the speed of light. These results will be of importance for the companion paper [15] (namely to clarify misconceptions on the relationship between gravitomagnetism and the curvature invariants discussed below).

As discussed above, this is a cancellation of translational (which comes as combination of Thomas and geodetic precessions) and rotational gravitomagnetism (which contributes as the Lense-Thirring precession). Another problem which in this spirit would also arise as a natural candidate to be the analogue of the electromagnetic problem of the previous section would be considering gyroscopes in geodesic motion, and look for the velocity field for which (momentarily) they do not precess; this eliminates Thomas precession, and we would be effectively looking for a cancellation between the geodetic and Lense Thirring precessions. This is an equally interesting problem, but not the one that asymptotically matches the electromagnetic counterpart. Let us denote such velocity field by \vec{v} ; in the PN regime it can be computed from e.g. Eqs. (3.4.38) of [3], or (40.33b) of [31], setting therein $\vec{a} = 0$ and the PPN parameters $\gamma = \mu = 1$ (note also that $-\Omega_{\star}$ in our notation corresponds to $\dot{\Omega}$ in [3], and Ω in [31]). Solving for $\vec{v} = v^{\phi}\vec{e}_{\phi}$ the equation $\Omega_{\star} = 0$, we obtain an

asymptotic velocity $v' \xrightarrow{r \rightarrow \infty} 2a/3r = (4/3)v_{(\mathbf{B}=0)}$, which differs by a factor from (55) and (72).

Velocity for which $\mathbb{H}_{\alpha\beta} = 0$.— We have seen in Sec. IV B 1 that, by virtue of Eq. (I.3a) of Table I (i.e. the laws of electromagnetic induction), for a moving observer, $B_{[\alpha\beta]} \neq 0$; and thus $B_{\alpha\beta} \neq 0$, *always*. But we have seen also that, in the equatorial plane, there is a velocity field for which $B_{(\alpha\beta)} = 0$. Since the gravitomagnetic tidal tensor is symmetric in vacuum: $\mathbb{H}_{\alpha\beta} = \mathbb{H}_{(\alpha\beta)}$, we expect, in the spirit of the analogy, that $\mathbb{H}_{\alpha\beta} = 0$ in the analogous gravitational setup. We shall now find the velocity fields for which $\mathbb{H}_{\alpha\beta} = 0$. It turns out that this can be done by a procedure very similar to the problem of finding the velocity field $\vec{v}_{(\mathbf{B}=0)}$ for which the magnetic field B^α (not the tidal tensor $B_{\alpha\beta}$!) vanishes, due to the *formal* analogy between the decompositions of the Weyl and Maxwell tensors in electric and magnetic parts [11–13], and the quadratic invariants they form.

In vacuum the Riemann tensor becomes the Weyl (10 independent components), which can be irreducibly decomposed (see e.g. [13]), with respect to a unit timelike 4-vector u^α , in terms of two spatial tensors, the gravito-electric $\mathbb{E}_{\alpha\beta}$ and gravito-magnetic $\mathbb{H}_{\alpha\beta}$ tidal tensors:

$$R_{\alpha\beta}^{\gamma\delta} = 4 \left\{ 2u_{[\alpha} u^{\gamma]} + g_{[\alpha}^{\gamma]} \right\} (\mathbb{E}^u)_{\beta]}^{\delta]} + 2 \left\{ \epsilon_{\alpha\beta\mu\nu} (\mathbb{H}^u)^{\mu[\delta} u^{\gamma]} u^\nu + \epsilon^{\gamma\delta\mu\nu} (\mathbb{H}^u)_{\mu[\beta} u_{\alpha]} u_\nu \right\} \quad (74)$$

In vacuum, $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ are both symmetric and traceless, possessing 5 independent components each, thus encoding the 10 independent components of $R_{\alpha\beta\gamma\delta}$.

The decomposition (74) is *formally* analogous to the decomposition (1) of $F^{\alpha\beta}$ in terms of E^α and B^α . Moreover, the gravitational tidal tensors form the second order scalar invariants *formally* (see discussion in Sec. 4 of [14]) analogous to the electromagnetic invariants (51)-(52):

$$\mathbb{E}^{\alpha\gamma} \mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma} \mathbb{H}_{\alpha\gamma} = \frac{1}{8} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \equiv \frac{1}{8} \mathbf{R} \cdot \mathbf{R} \quad (75)$$

$$\mathbb{E}^{\alpha\gamma} \mathbb{H}_{\alpha\gamma} = \frac{1}{16} R_{\alpha\beta\gamma\delta} \star R^{\alpha\beta\gamma\delta} \equiv \frac{1}{16} \star \mathbf{R} \cdot \mathbf{R}, \quad (76)$$

which are usually combined into $I \equiv (\mathbf{R} \cdot \mathbf{R} + i \star \mathbf{R} \cdot \mathbf{R})/8$. However, these invariants are not sufficient¹² for an analysis like the one we did for the electromagnetic case. Whereas the invariants (51)-(52) are the only two independent scalar invariants of $F^{\alpha\beta}$, in the case of $R_{\alpha\beta\gamma\delta}$ (for vacuum) there also two independent *cubic* invariants, given by $A \equiv R^{\alpha\beta}_{\lambda\mu} R^{\lambda\mu}_{\rho\sigma} R^{\rho\sigma}_{\alpha\beta}/16$ and $B \equiv R^{\alpha\beta}_{\lambda\mu} R^{\lambda\mu}_{\rho\sigma} \star R^{\rho\sigma}_{\alpha\beta}/16$, and combined into $J \equiv A - iB$. It turns out (cf. [15, 75, 76]) that one obtains formally equivalent statements to (i)-(ii) of the electromagnetic

case, replacing \mathbf{F} by \mathbf{R} , *provided that the condition $\mathbb{M} \equiv I^3/J^2 - 6 \geq 0$ (real or infinite) is added to ii)*; that is: i) $\star \mathbf{R} \cdot \mathbf{R} \neq 0 \Rightarrow \mathbb{E}_{\alpha\gamma}$ and $\mathbb{H}_{\alpha\gamma}$ are both non-vanishing for all observers; ii) $\star \mathbf{R} \cdot \mathbf{R} = 0$, $\mathbf{R} \cdot \mathbf{R} > 0$, with $M \geq 0 \Rightarrow$ there are observers for which $\mathbb{H}_{\alpha\gamma}$ vanishes (“Purely Electric” spacetime)¹³. Further details and comments on this classification shall be given in [15].

For the case of Kerr spacetime, which is of Petrov type D, the condition $\mathbb{M} \geq 0$ (real) is satisfied, since $I^3 = 6J^2$ (see e.g. [76]). Thus one only has to worry about the invariants (75)-(76), which have the structure (the exact expressions are presented in [15]):

$$\begin{cases} \mathbb{E}^{\alpha\gamma} \mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma} \mathbb{H}_{\alpha\gamma} \approx \frac{6M^2}{r^6} > 0 \\ \mathbb{E}^{\alpha\gamma} \mathbb{H}_{\alpha\gamma} \approx \frac{18JM \cos \theta}{r^7} (= 0 \text{ in the equatorial plane}) \end{cases}$$

formally analogous to its electromagnetic counterpart (54). Note in particular that the result $\mathbb{E}^{\alpha\gamma} \mathbb{H}_{\alpha\gamma} = 0$ for the equatorial plane ($\theta = \pi/2$) is *exact*. This means that *in the equatorial plane, there are observers for which the magnetic tidal tensor vanishes*. It is straightforward to determine the 4-velocity of such observers. In the equatorial plane, the non-zero components of the gravito-magnetic tidal tensor $\mathbb{H}_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\nu} u^\mu u^\nu$ seen by an arbitrary observer of 4-velocity $u^\alpha = (u^t, u^r, u^\theta, u^\phi)$, are given *exactly* by ($\alpha \equiv 3M/r^3$):

$$\begin{aligned} \mathbb{H}_{r\theta} &= \alpha \left[(2a^2 + r^2) u^\phi u^t - a(a^2 + r^2) (u^\phi)^2 - a(u^t)^2 \right] \\ \mathbb{H}_{r\phi} &= \alpha (a^2 + r^2) (au^\phi - u^t) u^\theta \\ \mathbb{H}_{rt} &= \alpha a (au^\phi - u^t) u^\theta \\ \mathbb{H}_{\theta\phi} &= \alpha a [(a^2 + r^2) u^\phi - au^t] u^r \\ \mathbb{H}_{\theta t} &= -\alpha [(a^2 + r^2) u^\phi - au^t] u^r \\ \mathbb{H}_{\phi\phi} &= -2\alpha a u^r u^\theta = \mathbb{H}_{tt}; \quad \mathbb{H}_{\phi t} = \alpha (2a^2 + r^2) u^r u^\theta \end{aligned} \quad (77)$$

Then it is straightforward to show that in order to make all components $\mathbb{H}_{\alpha\beta}$ vanish, we must have $u^\theta = 0$ (i.e., the observer must move in the equatorial plane, as expected and in analogy with the electromagnetic example above) and

$$v^\phi = \frac{u^\phi}{u^t} = \frac{a}{a^2 + r^2} \equiv v_{(\mathbb{H}=0)}^\phi \quad (78)$$

Thus, observers with angular velocity $v^\phi = v_{(\mathbb{H}=0)}^\phi$ see a vanishing gravito-magnetic tidal tensor in the equatorial plane. Again, no restriction is imposed on u^r , apart from the normalization condition $u^\alpha u_\alpha = -1$. The velocity

¹² We thank L. Wylleman for his input on this issue.

¹³ The case $\star \mathbf{R} \cdot \mathbf{R} = 0$, $\star \mathbf{R} \cdot \mathbf{R} < 0$ would mean that there would be observers for which $\mathbb{E}_{\alpha\gamma}$ vanishes (but no “Purely Magnetic” vacuum solutions are known, and it has been conjectured that do not exist, see e.g. [77, 78]).

field corresponding to the case $u^r = 0$ is plotted in Fig. 2b. It is interesting to note that, asymptotically, this field matches the velocity (58) for which the *symmetric part* of the magnetic tidal tensor $B_{\alpha\beta}$ vanishes in the electromagnetic analogue (and, up to a factor of 2, the velocity (72) for which the precession vanishes).

As discussed above, the velocity field (78) has no electromagnetic analogue. The magnetic tidal tensor $B_{\alpha\beta}$ can never vanish for a moving observer, due to Maxwell's (vacuum) equation $\nabla \times \vec{B} = \partial \vec{E} / \partial t$; or, in other words, due to the laws of electromagnetic induction. The fact that in the gravitational case one can find a velocity field for which $\mathbb{H}_{\alpha\beta} = 0$ is an illustration of the physical consequences of the different symmetries of $\mathbb{H}_{\alpha\beta}$ compared to $B_{\alpha\beta}$, and means that there is no gravitational analogue to the curl of \vec{B} induced by a time-varying field. In other words, it signals the absence of electromagnetic-like induction effects in gravity.

Note that these fundamental differences between the two interactions are manifest even in the weak field and slow motion regime, for taking the field to be weak, either by going far away from the source, or by taking a to be small, that only amounts to make the velocity for which F_G^α vanishes (by contrast with F_{EM}^α) smaller, since $|v| \approx a/r$. That illustrates how misleading can be the usual treatments in the literature (e.g. [2]) concerning the topic of “gravitoelectromagnetism” in the framework of the linearized theory, which naively cast the force on a gyroscope in an expression of the type $\vec{F}_G = -\nabla(\vec{S} \cdot \vec{B}_G)$, identical to the electromagnetic force on a magnetic dipole. As discussed in Sec. III, the similarity between the two forces occurs only in the very special case that the gyroscope's center of mass is at rest relative to the source.

No circular geodesics for spinning material particles in Kerr spacetime — the vanishing of F_G^α for gyroscopes carried by a congruence of observers of the form (78) makes us wonder if a spinning particle can move in circular geodesics around a Kerr Black Hole, which we will now check. Since the observers (78) have *prograde motion* with angular velocity v^ϕ , we are only interested in checking prograde circular geodesics, whose angular velocity reads (e.g. [79]):

$$v_{\text{geo}}^\phi \equiv \frac{u_{\text{geo}}^\phi}{u_{\text{geo}}^t} = \frac{1}{a + \sqrt{r^3/M}}. \quad (79)$$

Equating this expression to (78), we obtain

$$\frac{a}{a^2 + r^2} = \frac{\sqrt{M}}{r^{3/2} + a\sqrt{M}} \Leftrightarrow r = \frac{a^2}{M}.$$

Thus, for given M and J , at $r = a^2/M$ there *would be* circular orbits along which $\mathbb{H}_{\alpha\beta} = 0$. However such orbits are not possible since r does not lie outside the horizon. $r_+ = M + \sqrt{M^2 - a^2}$, thus the condition $r \geq r_+$ implies

$$\frac{a^2}{M} \geq M + M\sqrt{1 - a^2/M^2} \Leftrightarrow 1 - A^2 \leq -\sqrt{1 - A^2}$$

where we defined the dimensionless parameter $A \equiv a/M$. $A = 1$ is the extreme Kerr case; $A > 1$ is a naked singularity; thus, excluding the naked singularity scenario, the circular orbit would exist only in the extreme case, it would be precisely at the horizon, and thus (apart from being obviously unstable), it would be a null geodesic.

Finally, it is interesting to note that the velocity field (78) appeared independently, in a completely different context (not related to tidal effects), in a recent paper, see Eq. (30) of [80]. Therein it is shown that the Kerr-Newman metric can be obtained by a simple rescaling of an orthonormal tetrad field in Minkowski space, constructed from spheroidal coordinates in differential rotation, each spheroidal shell $r = \text{constant}$ rotating rigidly. It turns out that the angular velocity of the shells is precisely $v^\phi = v_{(\mathbb{H}=0)}^\phi$, both in Minkowski space and in the Kerr-Newman metric (with respect to the static observers); that is, in the light of our discussion above, the gyroscopes comoving with the shells measure vanishing magnetic curvature $\mathbb{H}_{\alpha\beta}$ and feel no force: $F_G^\alpha = 0$.

3. Circular geodesics in Kerr de Sitter Spacetimes

The failure to obtain circular geodesics for material spinning particles in the previous section was due to the fact that the angular velocity of circular geodesics in the Kerr spacetime dies off as $r^{2/3}$, whereas the angular velocity for which $\mathbb{H}_{\alpha\beta} = 0$ dies off as r^{-2} ; in other words, “geodesics are too fast”. But they should be possible in other spacetimes; in this spirit, Kerr de Sitter come as natural candidate, since a repulsive Λ should “slow down” the circular geodesics. Next we will show that this is indeed the case. In Boyer-Lindquist coordinates, the Kerr-de Sitter metric takes the form

$$ds^2 = -\frac{\Delta_r}{\chi^2 \Sigma} (dt - a \sin^2 \theta d\phi)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\chi^2 \Sigma} [adt - (a^2 + r^2)d\phi]^2, \quad (80)$$

where

$$\begin{aligned} \Delta_r &\equiv r^2 - 2Mr + a^2 - \frac{\Lambda}{3} r^2 (r^2 + a^2) \\ \Delta_\theta &\equiv 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta \\ \chi &\equiv 1 + \frac{\Lambda}{3} a^2 \\ \Sigma &\equiv r^2 + a^2 \cos^2 \theta \end{aligned} \quad (81)$$

First one notices that since $\Lambda \neq 0 \Rightarrow R_{\mu\nu} \neq 0$ the classification based on scalar invariants used in the previous section does not apply to herein the Riemann tensor. Such classification still holds however *generically* for the Weyl tensor $C_{\alpha\mu\beta\nu}$, see e.g. [76]. And since Λ does not generate mass currents, we have $\mathbb{H}_{\alpha\beta} = \mathbb{H}_{(\alpha\beta)} = \mathcal{H}_{\alpha\beta}$, denoting $\mathcal{H}_{\alpha\beta} \equiv \star C_{\alpha\mu\beta\nu} U^\mu U^\nu$ the magnetic part of the Weyl tensor. Thus, solving for $\mathbb{H}_{\alpha\beta} = 0$ amounts to check the

conditions under which $\mathcal{H}_{\alpha\beta}$ vanishes, which reduces to the same procedure of the previous section, but this time using the invariants of the Weyl tensor. The invariants have a similar structure, similarly leading to the conclusion that in the equatorial plane, there are observers for which $\mathcal{H}_{\alpha\beta} = \mathbb{H}_{\alpha\beta} = 0$. Actually, the magnetic tidal tensor $\mathbb{H}_{\alpha\beta}$ for the metric (80) is generically obtained by simply multiplying expressions (77) by $9/(3 + a^2\Lambda)^3$:

$$(\mathbb{H}_{\text{Kerr-dS}})_{\alpha\beta} = \frac{9}{(3 + a^2\Lambda)^3} (\mathbb{H}_{\text{Kerr}})_{\alpha\beta}.$$

Thus, the velocity field for which $\mathbb{H}_{\alpha\beta} = 0$ is given by the same Eq. (78).

Now we need to check if this velocity field can correspond to a family of circular geodesics. We can very easily derive the geodesic equations from the Euler-Lagrange equations

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \quad (82)$$

with Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu. \quad (83)$$

To compute the circular geodesics we only need the r -equation,

$$\frac{d}{d\lambda} (g_{rr} \dot{r}) = \frac{1}{2} g_{\mu\nu, r} \dot{x}^\mu \dot{x}^\nu, \quad (84)$$

which for circular trajectories ($\dot{\theta} = 0$, $\dot{r} = 0 = \ddot{r}$) reduces to

$$g_{\phi\phi, r} (v_{\text{geo}}^\phi)^2 + 2g_{t\phi, r} v_{\text{geo}}^\phi + g_{tt, r} = 0, \quad (85)$$

where $v_{\text{geo}}^\phi \equiv \frac{\dot{\phi}}{\dot{t}}$ is the angular velocity of the geodesics. Solving this equation in the equatorial plane $\theta = \frac{\pi}{2}$ gives us

$$(v_{\text{geo}}^\phi)_\pm = \frac{-Ma + \frac{\Lambda}{3} ar^3 \pm \sqrt{Mr^3 - \frac{\Lambda}{3} r^6}}{r^3 - a^2 M + \frac{\Lambda}{3} a^2 r^3}, \quad (86)$$

which reduces to standard Kerr case, Eq. (79), when $\Lambda = 0$.

There are two things we need to check, though. The first is that the geodesics lie outside the black hole event horizon (and inside the cosmological horizon), the second is that the geodesics are time-like. The horizons are located at the real roots of $\Delta_r = 0$, which gives us the equation

$$r^2 - 2Mr + a^2 - \frac{\Lambda}{3} r^2 (r^2 + a^2) = 0. \quad (87)$$

To guarantee that the geodesic is time-like we obviously have to make sure that $ds^2 < 0$, which for our case gives the following condition

$$\begin{aligned} -r^2 + 2Mr(1 - v_{\text{geo}}^\phi a)^2 + r^2 (v_{\text{geo}}^\phi)^2 (a^2 + r^2) \\ + \frac{\Lambda}{3} r^2 (r^2 + a^2) (1 - v_{\text{geo}}^\phi a)^2 < 0. \end{aligned} \quad (88)$$

To find spinning particles that follow circular geodesics, we have only to equate the *prograde* solutions of Eq. (86) to (78),

$$\frac{a}{a^2 + r^2} = \frac{-Ma + \frac{\Lambda}{3} ar^3 + \sqrt{Mr^3 - \frac{\Lambda}{3} r^6}}{r^3 - a^2 M + \frac{\Lambda}{3} a^2 r^3}. \quad (89)$$

Unfortunately, we can no longer analytically solve this equation for r in general, but for our present purposes it suffices to numerically show that such an r exists for some particular cases of a and Λ . Consider for example the case $a/M = 0.8$, $\Lambda M^2 = 0.001$. Solving equation (89) for r , we find, as the only acceptable solution $r \simeq 14.2025M$ (the other roots are either complex or fall within the horizon). We can easily see from equations (87) and (88) that this geodesic is both time-like and outside the event horizon. Obviously, several other solutions of (89) for different values of a and Λ are possible. We generically find that, for fixed a/M , decreasing values of ΛM^2 correspond to solutions of (89) with increasing values of r . These solutions seem to be all unstable, however.

Sec. IV in brief — the physical consequences of the different symmetries of $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$

- In electromagnetism, due to vacuum Eq. $B_{[\alpha\beta]} = \star F_{\alpha\beta\gamma} U^\gamma / 2$, a force $F_{EM}^\alpha = B^{\alpha\beta} \mu_\beta$ is exerted on the dipole *whenever it moves* in an inhomogeneous field (except for very special orientations of $\vec{\mu}$).
- In gravity, $\mathbb{H}_{[\alpha\beta]} = 0$, and there are velocity fields for which $\mathbb{H}_{\alpha\beta} = 0$, i.e., for which *gyroscopes feel no force*:
 - in the examples studied, they correspond to the situations where $B_{\alpha\beta} = B_{[\alpha\beta]}$ in the electromagnetic analogue;
 - there are even geodesic motions for spinning particles: radial geodesics in Schwarzschild, circular geodesics in Kerr-dS spacetimes.
- Formal analogy between the quadratic scalar invariants of $R_{\alpha\beta\gamma\delta}$ and $F_{\alpha\beta}$ are useful to obtain velocities for which $\mathbb{H}_{\alpha\beta} = 0$.

V. TIME PROJECTIONS OF THE FORCES, AND WORK DONE ON THE TEST PARTICLE

A fundamental difference between the gravitational and electromagnetic interactions concern the time projections of the forces F_G^α and F_{EM}^α in the different frames, which we shall explore in this section. We start by explaining the meaning of the time projection of a force in

a given frame, and its relation with the work done by it and the particle's energy.

Consider a congruence of observers $\mathcal{O}(u)$ with 4-velocity u^α , and let U^α denote the 4-velocity of a test particle. The following relation generically holds [9]:

$$U^\alpha = \gamma(u^\alpha + v^\alpha); \quad \gamma \equiv -u^\alpha U_\alpha = \frac{1}{\sqrt{1 - v^\alpha v_\alpha}} \quad (90)$$

where $v^\alpha = U^\alpha/\gamma - u^\alpha$ is the velocity of the test particle relative to the observers $\mathcal{O}(u)$; in the frame $u^i = 0$, v^i is the ordinary 3-velocity. The energy of the test particle relative to $\mathcal{O}(u)$ is $E \equiv -P^\alpha u_\alpha$, and its rate of change (the “power equation”)

$$\frac{dE}{d\tau} = -F^\alpha u_\alpha - P^\alpha u_{\alpha;\beta} U^\beta, \quad (91)$$

where $F^\alpha \equiv DP^\alpha/d\tau$ denotes the 4-force. Thus we see that the variation energy of the particle relative to $\mathcal{O}(u)$ consists of two terms: the time projection of F^α along u^α , plus a term depending on the variation of u^α along the test particle worldline. The first term is interpreted as the rate of work, *as measured by* $\mathcal{O}(u)$, done by the force on the test particle. Using $u^\alpha = U^\alpha/\gamma - v^\alpha$, we can write it as

$$-F^\alpha u_\alpha = -\frac{F^\alpha U_\alpha}{\gamma} + F^\alpha v_\alpha. \quad (92)$$

In order to better explain the meaning of this expression, let us start by the simplest case, a point particle with no internal structure (in our expansion, a monopole particle). In this case the momentum is parallel to the 4-velocity, $P^\alpha = mU^\alpha$, and its mass is a constant, $m = m_0$; hence the force is parallel to the acceleration: $F^\alpha \equiv DP^\alpha/d\tau = m_0 a^\alpha$. Since U^α is always orthogonal to a^α (this is a general statement, which follows from the normalization condition $U^\alpha U_\alpha = -1$), that implies $F^\alpha U_\alpha = 0$; i.e., the time projection of F^α , *in the particle's frame*, vanishes. Such force is said to be *spatial* with respect to U^α , just like the acceleration. That leads to the expression: $-F^\alpha u_\alpha = F^\alpha v_\alpha$, telling us that the time-projection of the 4-D force F^α is the familiar power (i.e., the rate of work per unit of proper time τ) transferred to the particle by the 3-D force $(h^U)^\alpha F^\mu$ (see e.g. [35, 49]). Consider also for simplicity an inertial frame, so that the second term of (91) vanishes; then $-F^\alpha u_\alpha = dE/d\tau = m_0 d\gamma/d\tau$, i.e., $F^\alpha v_\alpha = m_0 d\gamma/d\tau$ is the rate of variation of kinetic energy of translation of the particle's CM. It is clear in particular that $dE/d\tau = F^\alpha v_\alpha = 0$ in a frame comoving with the test particle, since therein $v^\alpha = 0$. This is the type of force we are more familiar with; an example is the Lorentz force on a charged particle, $DP^\alpha/d\tau = qF^{\alpha\beta}U_\beta$, whose projection along u^α reads $-u^\alpha DP_\alpha/d\tau = \gamma q v^\alpha E_\alpha$, yielding the rate of work (per unit proper time) done by the electric field on the particle moving with velocity v^α relative to $\mathcal{O}(u)$.

However, if the particle has internal structure, its internal degrees of freedom may store energy, which in general

will be exchanged with the energy of the external fields and the kinetic energy of the center of mass. Therefore the proper mass of the particle $m = -P^\alpha U_\alpha$ no longer has to be a constant, cf. Sec. II E. (For the case of a quasi-rigid spinning particle, its internal energy is essentially kinetic energy of rotation about the center of mass, which varies in the presence of an electromagnetic field, as explained in Sec. VI below, see also [65–67].) Also the momentum will not be parallel to U^α , as the particle in general possesses hidden momentum, cf. Sec. II D. These, together (as we shall see next), endow F^α with a nonvanishing time projection: $F^\alpha U_\alpha \neq 0$. Also the contribution $F^\alpha v_\alpha$ to (92) is no longer solely a variation of translational kinetic energy, see e.g. Eq. (103) below.

Let us turn our attention now to the second term of Eq. (91). Decomposing (e.g. [9, 13, 34])

$$u_{\alpha;\beta} = -a(u)_\alpha u_\beta + \omega_{\alpha\beta} + \theta_{\alpha\beta} \quad (93)$$

where $a(u)^\alpha = u^\alpha_{;\beta} u^\beta$ is the acceleration of $\mathcal{O}(u)$ (not the particle's!), $\omega_{\alpha\beta} \equiv (h^u)^\lambda_\alpha (h^u)^\nu_\beta u_{[\lambda;\nu]}$ the vorticity, and $\theta_{\alpha\beta} \equiv (h^u)^\lambda_\alpha (h^u)^\nu_\beta u_{(\lambda;\nu)}$ the *total* shear of the congruence ($\theta_{\alpha\beta} \equiv \sigma_{\alpha\beta} + \theta(h^u)_{\alpha\beta}/3$, being $\sigma_{\alpha\beta}$ as usual the traceless shear and θ the expansion scalar). Let us denote by $G(u)^\alpha = -a(u)^\alpha$ the “gravitoelectric field” [9, 34] measured in the frame $u^i = 0$. Decomposing $P^\alpha = mU^\alpha + P^\alpha_{\text{hid}}$, cf. Eq. (26), and U^α using (90), the second term of Eq. (91) becomes:

$$\begin{aligned} -P^\alpha u_{\alpha;\beta} U^\beta &= m\gamma^2 [G(u)_\alpha - \theta_{\alpha\beta} v^\beta] v^\alpha \\ &\quad + \gamma P^\alpha_{\text{hid}} [G(u)_\alpha - (\omega_{\alpha\beta} + \theta_{\alpha\beta}) v^\beta]. \end{aligned} \quad (94)$$

This part of $dE/d\tau$ depends only on the kinematical quantities of the congruence. That is, unlike the term (92), which arises from the 4-force F^α , the term (94) does not depend on any physical quantity one can *locally* measure; it is locally an artifact of the reference frame, which can always be made to vanish by choosing a locally inertial one. Its importance (in a non-local sense) should not however be overlooked. To understand it, consider a simple example, a monopole particle in Kerr spacetime, from the point of view of the congruence of *static observers* (i.e., u^α parallel to the time-like Killing vector field $\xi \equiv \partial/\partial t$, in Boyer Lindquist coordinates). Since the congruence is rigid, $\theta_{\alpha\beta} = 0$; also, for a monopole particle, $P^\alpha_{\text{hid}} = 0$, and, in a gravitational field, $F^\alpha = 0$ (the particle moves along a geodesic). Therefore, the energy variation reduces to $dE/d\tau = -P^\alpha u_{\alpha;\beta} U^\beta = m\gamma^2 G(u)_\alpha v^\alpha$; which is the rate of “work” (per unit proper time τ) done by the gravitoelectric “force” [5, 9, 34] $m\gamma^2 G(u)^\alpha$. (In the Newtonian limit, reduces to the work of the Newtonian force $m\vec{G}$.) Hence we see that (94) is the part of (91) that encodes the change in translational kinetic energy of a particle (relative to static observers) which occurs due to the gravitational field, without the action of any (physical, covariant) force, i.e., for particles in geodesic motion.

Substituting Eqs. (92) and (94) in (91), we obtain a generalization, for the case of test particles with varying m and hidden momentum, of the “power” equation (6.12) of [9] (the latter applying to monopole particles only).

A. Time components in test particle’s frame

One fundamental difference between the tensorial structure of $\mathbb{H}_{\alpha\beta}$ and $B_{\alpha\beta}$ is that whereas the former is a spatial tensor in both indices: $\mathbb{H}(u)_{\alpha\beta}u^\beta = \mathbb{H}(u)_{\alpha\beta}u^\alpha = 0$ (this follows from the symmetries of the Riemann tensor), the latter is not: $B_{\alpha\beta}(u)u^\alpha = 0$ but $B_{\alpha\beta}(u)u^\beta = \star F_{\alpha\gamma;\beta}u^\gamma u^\beta \neq 0$ in general. That means that whereas F_G^α is a spatial force, F_{EM}^α has a non vanishing time projection in the particle’s proper frame, $-F_{EM}^\alpha U_\alpha$. Let us discuss its physical meaning. First note, from Eq. (I.4a), that

$$F_{EM}^\alpha U_\alpha = B^{\beta\alpha} U_\alpha \mu_\beta = \epsilon_{\beta\delta\mu\nu} U^\delta E^{[\mu\nu]} \mu^\beta; \quad (95)$$

where we see that it consists on a coupling between μ^α and the space projection of the antisymmetric part of the electric tidal tensor $E_{\alpha\beta}$ measured in the particle’s frame, which, as discussed in Sec. II B (see also [6, 34]), encodes Faraday’s law of induction. Indeed, for an arbitrarily accelerated frame (non-rotating and non-shearing) we may replace $E_{\alpha\beta}$ by the covariant derivative of the electric field $E_{\alpha;\beta}$, cf. Eq. (20), and Eq. (95) becomes, in vector notation, $F_{EM}^\alpha U_\alpha = -(\nabla \times \vec{E}) \cdot \vec{\mu}$. Its significance becomes clear if one thinks about a magnetic dipole as a small current loop of area A and magnetic moment $\vec{\mu} = \vec{n}AI$, see Fig. 3a. It then follows:

$$-F_{EM}^\alpha U_\alpha = (\nabla \times \vec{E}) \cdot \vec{n}AI = I \int_{S^2} \mathbf{dE} = I \oint_{\text{loop}} \vec{E} \equiv \mathcal{P}_{\text{ind}} \quad (96)$$

where \mathbf{dE} is the 3-D 2-form $\mathbf{dE} \equiv \frac{1}{2} E_{j;i} \mathbf{dx}^i \wedge \mathbf{dx}^j$, and \vec{E} is the *induced electric field*, coming from the generalized induction¹⁴ law (21). In the second equality we used the fact that the loop is (by definition) infinitesimal, the third equality is the application of Stokes theorem in the 3-D rest space of the dipole.

Thus $-F_{EM}^\alpha U_\alpha \equiv \mathcal{P}_{\text{ind}}$ is the rate of work transferred to the dipole by Faraday’s law of induction. Using Eqs. (31) and (32), we see that it consists in the variation of the proper mass m , *minus* the projection along U^α of the derivative of the hidden momentum (to which only the

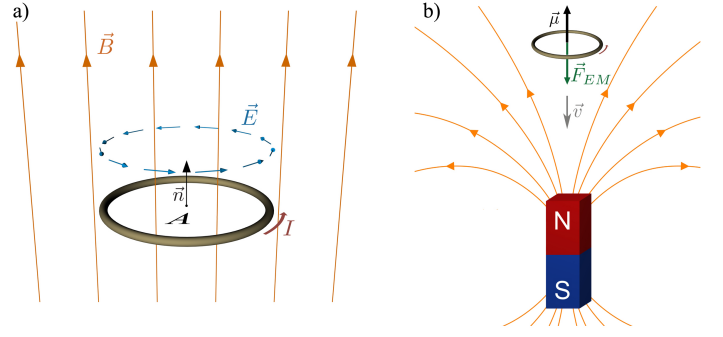


Figure 3: A magnetic dipole (depicted as a current loop) in the inhomogeneous magnetic field of a strong magnet, from the point of view of two different frames: a) the particle’s rest frame; b) the frame of the static observers (at rest relative to the strong magnet). $\vec{\mu} = IA\vec{n}$ (magnetic moment); $A \equiv$ area of the loop; $I \equiv$ current through the loop; $\vec{n} \equiv$ unit vector normal to the plane of the loop; $\vec{E} \equiv$ induced electric field. In the dipole’s frame there is a non-vanishing work done on it by \vec{E} , at a rate $\mathcal{P}_{\text{ind}} = -F_{EM}^\alpha U_\alpha$, which is reflected in a variation of proper mass m . From the point of view of static observers u^α , the work is zero ($-F_{EM}^\alpha u_\alpha = \mathcal{P}_{\text{ind}} + \mathcal{P}_{\text{trans}} = 0$), manifesting that a stationary magnetic field does no work. That may be regarded as an exact cancellation between \mathcal{P}_{ind} and $\mathcal{P}_{\text{trans}}$.

“electromagnetic” hidden momentum contributes):

$$\mathcal{P}_{\text{ind}} = \frac{dm}{d\tau} - \frac{DP_{\text{hid}}^\alpha}{d\tau} U_\alpha = \frac{dm}{d\tau} - \frac{DP_{\text{hidEM}}^\alpha}{d\tau} U_\alpha. \quad (97)$$

Note also, from Eq. (31), that (since m has the interpretation of the *energy of the dipole as measured in its proper frame*), \mathcal{P}_{ind} is the variation of the dipole’s energy $E = -P_0$ as measured in a momentarily comoving inertial frame. It should be mentioned that the failure to notice that F_{EM}^α has a time projection, and that m varies, has led to the difficulties in [16, 17] to covariantly describe the force on a magnetic dipole (namely to the claim that no covariant description of such forces is this scheme is possible).

The induction phenomenon in (96) has no counterpart in gravity. Since $\mathbb{H}_{\alpha\beta}$ is a spatial tensor, we *always have*

$$F_G^\alpha U_\alpha = 0 \quad (98)$$

which means that the *energy* of the gyroscope, as measured in a momentarily comoving inertial frame, is constant. The proper mass m is also constant, since, in the gravitational case, $P^\alpha a_\alpha = -U_\alpha DP_{\text{hid}}^\alpha/d\tau$ is always zero (see also Sec. II E). Hence, we see that the spatial character of gravitational tidal tensors *precludes* induction effects analogous to the electromagnetic ones.

B. Time components as measured by static observers

With respect to an arbitrary congruence of observers u^α , the time projection of the force exerted on a magnetic

¹⁴ This generalization of Maxwell-Faraday law for accelerated frames is needed if one is to deal with the electric and magnetic fields measured in the test particle’s frame, which in general accelerates. One could instead base the analysis in the inertial frame *momentarily* comoving with it, as done in Sec. V of [6], where $\partial \vec{B}/\partial \tau = -\nabla \times \vec{E}$ holds; the two treatments are equivalent.

dipole is, cf. Eq. (92):

$$-F_{\text{EM}}^\alpha u_\alpha = -\frac{F_{\text{EM}}^\beta U_\beta}{\gamma} + F_{\text{EM}}^\alpha v_\alpha = \frac{\mathcal{P}_{\text{ind}}}{\gamma} + F_{\text{EM}}^\alpha v_\alpha \quad (99)$$

where, in accordance with discussion above, we identify $\mathcal{P}_{\text{ind}} = -F_{\text{EM}}^\beta U_\beta$ as the power transferred to the dipole by Faraday's induction, and $F_{\text{EM}}^\alpha v_\alpha$ is the power transferred by the 3-force $(h^U)^\alpha_\mu F_{\text{EM}}^\mu$ exerted on it. Consider now a congruence of observers u^α for which the fields are covariantly constant $F_{;\gamma}^{\alpha\beta} u^\gamma = 0$, which we dub in this context “static observers”¹⁵. The time projection of the force in this frame vanishes:

$$-(F_{\text{EM}})_\alpha u^\alpha \equiv -\frac{DP_\alpha}{d\tau} u^\alpha = \star F_{\alpha\beta;\gamma} U^\beta \mu^\alpha u^\gamma = 0. \quad (100)$$

That tells us that the total work done on the dipole, *as measured in this frame*, is zero. The scalar $E = -P_\alpha u^\alpha$ (the energy of the particle in the $u^i = 0$ frame) is thus a conserved quantity, and using Eq. (26), we can write it in the form

$$E = m + T + E_{\text{hid}} = \text{constant}, \quad (101)$$

where we dub $E_{\text{hid}} \equiv -P_{\text{hid}}^\alpha u_\alpha$ as the “hidden energy” (i.e., the time component of the hidden momentum in the $u^i = 0$ frame), and $T \equiv (\gamma - 1)m$ is the kinetic energy of translation of the center of mass (cf. e.g. [49] p. 70), as measured in this frame. The reason for the later denomination is seen taking the Newtonian regime, where $T \approx mv^2/2$. In the (very few, to the authors' knowledge) existing literature addressing this problem, the fact that the total work done on the particle is zero is in [39] justified with an exact cancellation of $dT/d\tau$ and $dm/d\tau$; or in [65–67], for the case of a spherical spinning charged body, between the variations of T and kinetic energy of rotation about the CM (as shown in Sec. VI A 2, for such body and if $P_{\text{hid}}^\alpha = 0$, $dm/d\tau$ is a variation of kinetic energy of rotation). Eq. (101) shows however that, in the general case, the energy exchange occurs between three parts, E_{hid} also having a role. A suggestive example are the bobbings of a particle with magnetic dipole moment orbiting a cylindrical charge considered in Sec. III.B.1 of [29] (see also [25]).

In this work we are especially interested in the case: $P_{\text{hid}}^\alpha = 0$ ($\Rightarrow E_{\text{hid}} = 0$); in this case, $m + T = \text{constant}$, i.e., the energy exchange, due to the action of the force

F_{EM}^α , occurs only between proper mass (i.e., internal energy) and translational kinetic energy. Also, $\mathcal{P}_{\text{ind}} = dm/d\tau$, and for static observers in flat spacetime $u_{\alpha;\beta} = 0$, so that the second term of (91) vanishes. We can thus write:

$$\frac{dE}{d\tau} = -F_{\text{EM}}^\alpha u_\alpha = \mathcal{P}_{\text{ind}} + \mathcal{P}_{\text{trans}} = 0 \quad (102)$$

where

$$\mathcal{P}_{\text{trans}} \equiv \frac{dT}{d\tau} = \left(\frac{1}{\gamma} - 1\right) \frac{dm}{d\tau} + F_{\text{EM}}^\alpha v_\alpha \quad (103)$$

is the rate of variation of translational kinetic energy, and the the third equality may be obtained by differentiating (101) and comparing with (99). An example is the problem depicted in Fig. 3b): a magnetic dipole falling along the symmetry axis of the field generated by a strong magnet. $P_{\text{hid}}^\alpha = 0$ for this configuration. That this is a solution (a non-helical one) of the equations of motion supplemented with Mathisson-Pirani condition is shown by arguments analogous to the ones given in Appendix C 1 for the gravitational analogue depicted in Fig. 4a below. From the point of view of the static observers, $\vec{E} = 0$ and only magnetic field \vec{B} is present; since the latter does no work in any charge/current distribution, no work can be done on the dipole; thus naturally $F_{\text{EM}}^\alpha u_\alpha = 0$. According to Eq. (102), this arises from an exact cancellation between $\mathcal{P}_{\text{trans}}$ and \mathcal{P}_{ind} : on the one hand there is (in the $u^i = 0$ frame) an attractive spatial force F_{EM}^i causing the dipole to gain translational kinetic energy; on the other hand there is a variation of its internal energy (proper mass m) by induction, which allows for the total work to vanish (this is in agreement with rationale in [39], p. 21). See also in this respect Sec. VIC and Appendix B 4.

In gravity, since those induction effects are absent, cf. Eq. (98), such cancellation does not occur. For arbitrary u^α we have thus

$$-F_G^\alpha u_\alpha = F_G^\alpha v_\alpha \neq 0. \quad (104)$$

This implies in particular that, from the point of view of static observers, *by contrast with its electromagnetic counterpart*, F_G^α *does work* on the test particle (i.e., a stationary gravitomagnetic tidal field does work on mass currents). Let us discuss the implications. A conserved quantity for a spinning particle in a stationary spacetime is (e.g. [22, 24, 29, 81])

$$E_{\text{tot}} = -P^\alpha \xi_\alpha + \frac{1}{2} \xi_{\alpha;\beta} S^{\alpha\beta} = \text{constant} \quad (105)$$

where $\xi \equiv \partial/\partial t$ is the time-like *Killing vector field*. Consider the congruence of the static observers¹⁶ $\mathcal{O}(u)$, de-

¹⁵ The reason for this denomination is the fact that the condition $F_{;\gamma}^{\alpha\beta} u^\gamma = 0$ corresponds to the observers *at rest* with respect to the sources in the electromagnetic applications herein (the coulomb charge in Sec. IV A, the spinning charge of Secs. III and IV B or the magnet in Fig. 3b)). Note that for e.g. observers u^α in circular orbits around a Coulomb charge we have $F_{;\gamma}^{\alpha\beta} u^\gamma \neq 0$ (see expressions (57) and discussion therein), even though u'^α is in that case a symmetry of $F_{\alpha\beta}$: $\mathcal{L}_{u'} F_{\alpha\beta} = 0$, and the latter is explicitly time-independent in a rotating frame.

¹⁶ See point 8 of Sec. I D. In stationary asymptotically flat spacetimes, such as the Kerr metric studied below, these are observers rigidly fixed to the asymptotic inertial rest frame of the source. They are thus the closest analogue of the flat spacetime notion of observers at rest relative to the source in the electromagnetic systems above.

defined as the unit-time like vectors u^α tangent to ξ^α ; we may write $\xi^\alpha = \xi u^\alpha$, with $\xi \equiv \sqrt{-\xi^\alpha \xi_\alpha}$ (the lapse, or redshift factor, of the static observers, see e.g. [4]). The first term of (105), $-P^\alpha \xi_\alpha = E\xi$ is the “Killing energy”, a conserved quantity for the case of a non-spinning particle ($S^{\alpha\beta} = 0$) in geodesic motion, which yields its energy E with respect to the static observer at infinity¹⁷. It can be interpreted as the “total energy” (rest mass + kinetic + “Newtonian potential energy”) of a monopole particle in the gravitational field (e.g. [82]). The energy E_{tot} can likewise be interpreted as the energy at infinity for the case of a spinning particle. To see the interpretation of the second term in (105), $V \equiv \xi_{[\alpha;\beta]} S^{\alpha\beta}/2$, consider the case that $P_{\text{hid}}^\alpha = 0$. We have

$$0 = \frac{dE_{\text{tot}}}{d\tau} = -F_G^\alpha \xi_\alpha - m U^\alpha U^\beta \xi_{\alpha;\beta} + \frac{dV}{d\tau} \quad (106)$$

$$\Leftrightarrow \xi F_G^\alpha u_\alpha = \frac{dV}{d\tau} \quad (107)$$

where we used the Killing equation $\xi_{(\alpha;\beta)} = 0$. $-\xi F_G^\alpha u_\alpha = \xi F_G^\alpha v_\alpha$ is the rate work (per unit of particle’s proper time τ) of F_G^α , as measured by the static observers at infinity, and thus V is the spin-curvature potential energy associated with that work¹⁸. In order to compare with the electromagnetic Eq. (101), note that $d\xi/d\tau = -\gamma G(u)_\alpha v^\alpha$, and that for $P_{\text{hid}}^\alpha = 0$, $E = \gamma m = m + T$, thus we can expand $dE_{\text{tot}}/d\tau = d(\xi E + V)d\tau$ in the form

$$\xi \frac{dT}{d\tau} - \xi m \gamma^2 G_\alpha v^\alpha + \frac{dV}{d\tau} = 0. \quad (108)$$

The second term accounts for the “power” of the gravitoelectric “force” $m\gamma^2 \vec{G}(u)$ (which, as explained above, is *not* a physical force, it arises from the acceleration of the observer congruence, being non-zero even for geodesic motion); in the weak field limit this is the variation of Newtonian potential energy. Eq. (108) tells us that the variation of translational kinetic energy T comes from the potential energy V , and the power of $\vec{G}(u)$ (being m constant); this contrasts with the case of the magnetic dipole discussed above, where (also for $P_{\text{hid}}^\alpha = 0$) the kinetic energy comes from the variation of proper mass m , no potential energy being involved. In terms of the work done on the particle, F_G^α is thus more similar to the electromagnetic forces exerted on a monopole charge or on an electric dipole (for $Dd^\alpha/d\tau = 0$), see Appendix B 4, where the proper mass is as well constant and energy exchange is between T and potential energy.

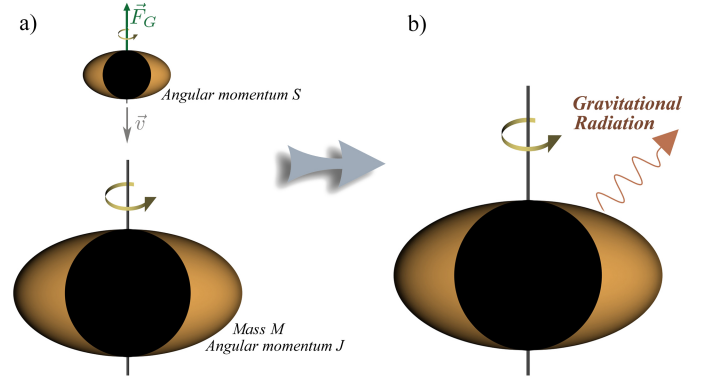


Figure 4: a) Gyroscope (small Kerr black hole) in the field of a large Kerr black hole; b) black hole merger. Evidence that, unlike its electromagnetic counterpart, gravitomagnetic tidal field *does* work: spin-dependent part of the energy released is the work (as measured by the static observers at infinity) of F_G^α .

There is a known consequence of the fact that F_G^α does work (and of the interaction energy V): the spin dependence of the upper bounds for the energy released by gravitational radiation (Fig. 4b) when two black holes collide, obtained by Hawking [30] from the area law.

In order to see that, consider the apparatus in Fig. 4: two Kerr black holes with spins aligned, a large one (mass M , spin $J = aM$) which is our source, and small one (4-velocity U^α , spin $S \equiv \sqrt{S^\alpha S_\alpha}$) which we take to be the test particle, falling into the former along the symmetry axis (How this is setup with Mathisson-Pirani condition is discussed in Appendix C 1). For axial fall, and given that \vec{S} is also along the axis, $P_{\text{hid}}^\alpha = P_{\text{hidI}}^\alpha = 0$. (Consider again the frame of the static observers $\mathcal{O}(u)$, $u^\alpha = \xi^\alpha/\xi$, i.e. the observers with zero 3-velocity in Boyer-Lindquist coordinates). For this configuration, V is a pure spin-spin potential energy (that one might also conclude from Sec. IV A, where we have seen that for $J = 0$, $F_G^\alpha = 0$); it reads:

$$V(r) = \pm \frac{2aMSr}{(a^2 + r^2)^2} = \int_\infty^{\tau(r)} \xi F_G^\alpha u_\alpha d\tau$$

the $+/-$ sign applying to the case that \vec{S} and \vec{J} are parallel/antiparallel. The second equality follows directly from Eq. (107), and can also be easily checked noting that, in Boyer-Lindquist coordinates, $\xi F_G^\alpha u_\alpha = (F_G)_0$, and computing explicitly the time component $(F_G)_0$ for axial fall, Eq. (37) of [1]. Thus we see that $V(r)$ is *minus* the work done by F_G^α as the particle goes from infinity to r . Let us comment on the presence of the lapse factor ξ in the integral above. Computing the work of F_G^α does not amount to integrate the power measured by the local static observers, Eq. (92): $-F_G^\alpha u_\alpha = F_G^\alpha v_\alpha$ (i.e., to sum up the work elements $dW \equiv F_G^\alpha v_\alpha d\tau$), as that would mean summing up energies measured by different observers; but to integrate instead the quantity $\xi F_G^\alpha v_\alpha$, which can be thought as summing work elements mea-

¹⁷ If the particle is in an orbit with always finite r , one can imagine this measurement process as follows: let $E_{\text{tot}}(\tau_1)$ be the total energy of the particle at τ_1 ; this means that if, at that instant, the particle was by some process converted into light, and sent to infinity, that radiation would reach infinity with an energy $E = -u^\alpha P_\alpha = E_{\text{tot}}(\tau_1)$.

¹⁸ One may check explicitly that $dV/d\tau = \xi_{\alpha;\beta\gamma} S^{\alpha\beta} U^\gamma = \xi F_G^\alpha v_\alpha$, noting that $DS^{\alpha\beta}/d\tau = 0$ if $P_{\text{hid}}^\alpha = 0$, and using the general relation for a Killing vector $\xi_{\mu;\nu\lambda} = R_{\lambda\sigma\mu\nu} \xi^\sigma$.

sured by static the observer *at infinity*. Let us now analyze the problem of the black hole merger. The increase of translational kinetic energy of the small black hole during the fall is given by Eq. (108) above (or by the power Eq. (91); note in this case that $\theta_{\alpha\beta} = 0$, since $\mathcal{O}(u)$ is a rigid congruence). The second term of (108) is the gain in kinetic energy due to the “Newtonian” attraction, and exists regardless of S^α ; V by its turn is a spin-spin energy; thus the kinetic energy of the particle (and therefore energy available to be released by gravitational radiation in the black hole merger) depends on S . Upper bounds for this energy which are, accordingly, spin dependent, were obtained in [30] by a totally independent method. From these limits, and for the case of the setup in Fig. 4, Wald [1] obtained an expression (Eq. (35) therein) for the amount of energy ΔE_s by which the upper bound is increased/reduced when \vec{S} is parallel/antiparallel to \vec{J} , comparing with the case $S = 0$ (fall along a geodesic). This energy is precisely *minus* the value of $V(r)$ at the horizon r_+ : $\Delta E_s = -V(r_+)$; that is, it is *the work done* by F_G^α on the small black hole as it comes from infinity to the horizon: $\Delta E_s = \int_{\infty}^{\tau(r_+)} (-\xi F_G^\alpha u_\alpha) d\tau$.

We close this section with some remarks on the meaning of the work done by the gravitomagnetic tidal field. To the frame of the static observers in the Kerr space-time one can associate a gravitomagnetic “vector” field \vec{H} (see Sec. II C, and e.g. [3, 5, 9, 34]; in the weak field regime this field is well known to be very similar to the electromagnetic counterpart, e.g. [2, 3, 83]) which causes inertial (i.e., fictitious) accelerations in test particles of the type $\vec{v} \times \vec{H}$, formally similar to the magnetic force $\vec{v} \times \vec{B}$. Namely the “force” is orthogonal to the velocity; hence this analogy might lead one to conclude that, similarly to its electromagnetic counterpart, the gravitomagnetic field can do no work on test particles. i) One must bear in mind that \vec{H} (by contrast with \vec{B}) has no local existence, it is a mere artifact of the reference frame; hence it would never be involved in a covariant quantity like the 4-force $DP^\alpha/d\tau$, or the work done by it; ii) both in electromagnetism and in gravity, *are the tidal fields that yield the force*; that is manifest in force Eqs. (I.1) of Table I. The electromagnetic tidal tensors are essentially derivatives of the fields; for this reason we were able to reason in terms of the fields in the applications depicted in Fig. 3 (even though are their derivatives that show up in the equations). But the gravitational tidal tensors cannot be cast as derivatives of the GEM fields, even in the weak field regime, except for very special conditions (as shown in Sec. III; see also [6, 34, 57]); the force F_G^α is thus in general very different from its electromagnetic counterpart. Namely it is so whenever the test particle moves relative to the source — so that the work of F_G^α can dramatically differ from that of F_{EM}^α , which is well exemplified by the contrast herein: in the test particle’s frame, we have $(F_{EM})_0 \neq 0$, $(F_G)_0 = 0$; in the frame of the static observers, we have precisely the *opposite* situation: $(F_{EM})_0 = 0$, $(F_G)_0 \neq 0$.

Sec. V in brief — the work done on the particle (magnetic dipole vs gyroscope)

- The time projection of the force, $-F^\alpha u_\alpha$, is the rate at which it does work on the particle, as measured by the observer of 4-velocity u^α .

Time projections along the particle’s worldline (U^α)

- The electromagnetic one is non-vanishing $F_{EM}^\alpha U_\alpha \neq 0$; it is the rate of work done by Faraday’s induction, arising from $E_{[\alpha\beta]}$ (or equivalently, from $B_{\alpha\beta} U^\beta$); reflected in a variation of m .
- The gravitational one is zero, $F_G^\alpha U_\alpha = 0$; the gyroscope’s proper mass m is constant; no analogue of Faraday’s induction (as $\mathbb{H}_{\alpha\beta} U^\beta = 0$).

Time projections relative to *static observers* (u^α)

- The electromagnetic one is zero, $F_{EM}^\alpha u_\alpha = 0$; a stationary electromagnetic field does no work on magnetic dipoles.
- The gravitational one is non-zero, $F_G^\alpha u_\alpha \neq 0$; gravitomagnetic tidal field does work — there is a spin-curvature potential energy; explains Hawking-Wald spin-spin interaction energy.

VI. BEYOND POLE-DIPOLE — THE TORQUE ON THE SPINNING PARTICLE

In our pole-dipole approximation, as we have seen in Sec. II C, it follows, from Eqs. (9) or (23), that for purely magnetic dipoles ($d^\alpha = 0$) if $\vec{\mu} = \sigma \vec{S}$, S^2 is a constant of the motion. This might be somewhat surprising. If one imagines the magnetic dipole as a spinning charged body, one would expect that, in a time-varying magnetic field, the induced electric field will in general exert a net torque on it, which will accelerate¹⁹ the rotation of the body. But in Eq. (23) we only find the term $\vec{\mu} \times \vec{B}$, coupling the field to the dipole moment (which is there in any case, even if the field is constant), and no term coupling to the *derivatives* of the electromagnetic fields; i.e., no trace of induction phenomena. Moreover we have seen in Secs. II E and V that the induced electric field does work on the spinning particle, causing a variation $dm/d\tau = -\vec{\mu} \cdot D\vec{B}/d\tau$ of its proper mass m ; that variation is shown in the non-relativistic treatments in [65, 67] (for

¹⁹ Unlike the torque due to the magnetic field, the torque due to the induced electric field will not be orthogonal to \vec{S} , and hence will in general change its magnitude. For instance, in the application in this section, \vec{E}_{ind} has circular lines around \vec{S} , so that $\vec{\tau}_{\text{ind}} \parallel \vec{S}$.

the case of a “rigid”, spherical body; it is not cast therein as a variation of proper mass m , but of the Hamiltonian contribution $-\vec{\mu} \cdot \vec{B}$ to be but a variation of its rotational kinetic energy. Thus we expect it as well to be associated to a variation of the spinning angular velocity, and hence of \vec{S} .

As we shall see below, this apparent inconsistency is an artifact inherent to the pole-dipole approximation, where terms $\mathcal{O}(R^2)$ (being R the size of the particle), which are of quadrupole type, are neglected; indeed, whereas the contribution of induction to the body’s energy is of the type $\vec{\mu} \cdot \vec{B}$, i.e., of dipole order, the associated torque involves *the trace* of the quadrupole moment of the charge distribution. And there is no analogous torque in the gravitational case, confirming the absence of an analogous gravitational effect.

For clarity, we will treat the two interactions (electromagnetic and gravitational) separately, and consider first, in Minkowski spacetime, a spinning charged body subject to an external electromagnetic field, and then in a gravitational field.

A. Electromagnetic torque

We will start by the electromagnetic case in flat spacetime. The equation for the spin evolution of an extended spinning charged body is, up to quadrupole order [21]:

$$\frac{D(S_{\text{can}})^{\alpha\beta}}{d\tau} = 2(P_{\text{Dix}})^{[\alpha}U^{\beta]} + 2Q^{\theta[\beta}F^{\alpha]}_{\theta} + 2m^{[\alpha}_{\rho\mu}F^{\beta]\mu;\rho} \quad (109)$$

where $(P_{\text{Dix}})^{\alpha}$ and $(S_{\text{can}})^{\alpha\beta}$ are defined by Eqs. (A4) and (A5) and consist on the sum of the *physical* momenta P^{α} , $S^{\alpha\beta}$ plus *electromagnetic terms* P'^{α} , $S'^{\alpha\beta}$; see Appendix A. $(S_{\text{can}})^{\alpha\beta}$ and $(P_{\text{Dix}})^{\alpha} + qA^{\alpha}$ are shown in [84] to be the canonical momenta associated to the Lagrangian of the system. $Q^{\alpha\beta}$ is the electromagnetic dipole moment defined in (7); $m^{\alpha\beta\gamma} \equiv 4Q^{(\alpha\beta)\gamma}/3$, being $Q^{\alpha\beta\gamma}$ an electromagnetic quadrupole moment defined as:

$$Q^{\alpha\beta\gamma} = \mathcal{J}^{\alpha[\beta\gamma]} + \frac{1}{2}q^{\alpha[\beta}U^{\gamma]} \quad (110)$$

where $\mathcal{J}^{\alpha\mu\nu}$ and $q^{\alpha\nu}$ are, respectively, the current and charge “quadrupole”²⁰ moments (cf. Eqs. (3.8)-(3.9) of

²⁰ Following the convention in some literature [29, 85], we are denominating expressions of the type (111)-(112) as “quadrupole moments”. Note however that it is also customary in the literature to denominate by charge quadrupole moment the traceless part of $q^{\alpha\beta}$. The former includes the trace and is non-zero even for a uniform chimerical body; the latter definition, more consistent with the actual picture of a quadrupole of charges, measures a certain type of deviation from spherical symmetry (and is zero for a uniform spherical body). $q^{\alpha\beta}$ may also be called the second moment of the charge, according to nomenclature in e.g. [31], p. 977.

[21]):

$$q^{\alpha\beta} \equiv \int_{\Sigma(\tau,U)} r^{\alpha} r^{\beta} j^{\gamma} d\Sigma_{\gamma} \quad (111)$$

$$\mathcal{J}^{\alpha\beta\nu} \equiv \int_{\Sigma(\tau,U)} r^{\alpha} r^{\beta} j^{\nu} w^{\gamma} d\Sigma_{\gamma} \quad (112)$$

The vector w^{γ} , defined as a vector such that displacement of every point by $w^{\gamma}d\tau$ maps $\Sigma(\tau)$ into $\Sigma(\tau+d\tau)$, is given in flat spacetime by Eq. (A1); it can be taken in def. (112) above as $w^{\gamma} \simeq U^{\gamma}$, since in this approximation we are neglecting terms up to r^3 . Hence we may write:

$$\mathcal{J}^{\alpha\beta\nu} = q^{\alpha\beta}U^{\nu} + \mathcal{J}^{\alpha\beta\gamma}(h^U)_{\gamma}^{\nu} \quad (113)$$

Using Eq. (A8ii) we may re-write Eq. (109) explicitly in terms of the physical angular momentum $S^{\alpha\beta}$:

$$\frac{DS^{\alpha\beta}}{d\tau} = \frac{D(S_{\text{can}})^{\alpha\beta}}{d\tau} - \frac{DS'^{\alpha\beta}}{d\tau}; \quad S'^{\alpha\beta} = F^{[\alpha}_{\sigma} q^{\beta]\sigma} \quad (114)$$

Note that $DS'^{\alpha\beta}/d\tau$ is a quadrupole type contribution.

We are interested in the torques τ^{α} , i.e., the vector that measures the rate of deviation of the spin vector from Fermi-Walker transport:

$$\tau^{\alpha} \equiv \frac{D_F S^{\alpha}}{d\tau} \Rightarrow \tau^{\sigma} = \frac{1}{2}\epsilon_{\alpha\beta}^{\sigma\delta} U_{\delta} \frac{DS^{\alpha\beta}}{d\tau}; \quad (115)$$

thus using Eqs. (109), (114) it follows:

$$\frac{D_F S^{\alpha}}{d\tau} = \tau_{\text{DEM}}^{\alpha} + \tau_{\text{QEM}}^{\alpha} \quad (116)$$

$$\tau_{\text{DEM}}^{\sigma} \equiv \epsilon^{\sigma}_{\alpha\beta\nu} U^{\nu} (d^{\alpha} E^{\beta} + \mu^{\alpha} B^{\beta}) \quad (117)$$

$$\tau_{\text{QEM}}^{\sigma} \equiv \tau_{\text{QEMcan}}^{\sigma} + \tau'^{\sigma} \quad (118)$$

$$\tau_{\text{QEMcan}}^{\sigma} \equiv m^{[\alpha}_{\rho\mu} F^{\beta]\mu;\rho} \epsilon^{\sigma}_{\alpha\beta} U_{\delta} \quad (119)$$

$$\begin{aligned} \tau'^{\sigma} &= \frac{1}{2}\epsilon^{\lambda}_{\alpha\beta} E^{[\alpha\beta]} (q^{\sigma}_{\lambda} - q^{\gamma}_{\lambda} \delta^{\sigma}_{\gamma}) \\ &\quad - \frac{1}{2}\epsilon^{\sigma}_{\alpha\beta} F^{\alpha}_{\gamma} \frac{Dq^{\beta\gamma}}{d\tau} \end{aligned} \quad (120)$$

$\tau_{\text{DEM}}^{\alpha}$ is the dipole torque already present in Eq. (9), i.e., just a covariant form for the familiar torque $\vec{\tau} = \vec{\mu} \times \vec{B} + \vec{d} \times \vec{E}$, where no trace of electromagnetic induction is found. We split the quadrupole torque $\tau_{\text{QEM}}^{\sigma}$ in two parts: $\tau_{\text{QEMcan}}^{\sigma}$, that we may dub the “canonical EM quadrupole torque”, is the torque²¹ coming from the third term of (109) (i.e. the electromagnetic quadrupole contribution to $D(S_{\text{can}})^{\alpha\beta}/d\tau$). The other part is $\tau'^{\sigma} \equiv$

²¹ Note that $\tau_{\text{QEMcan}}^{\sigma}$ is actually what in most literature is portrayed as the quadrupole torque, see e.g. [29]. It is clear however from Eq. (116) above (written in terms of the physical angular momentum $S^{\alpha\beta}$, S^{α}), that it is not the *total* quadrupole torque $\tau_{\text{QEM}}^{\sigma}$, and the discussion herein emphasizes how crucial this distinction is.

$-\frac{1}{2}\epsilon_{\alpha\beta}^{\sigma\delta}U_\delta DS'^{\alpha\beta}/d\tau$, and plays a crucial role in this discussion, as the first term of (120) is the torque due to the *electric field induced* in the CM frame, by the generalized Maxwell-Faraday law (21). This is what we shall now see. For that it is convenient to work in the Lorentz frame momentarily comoving with the particle's CM. In this frame the torque (about the center of mass) due to the induced electric field is $\vec{\tau}_{\text{ind}} = \int \rho_c \vec{r} \times \vec{E}_{\text{ind}} d^3x$, where $\rho_c \equiv -j^\alpha U_\alpha$ is the charge density in the CM frame. Let us expand \vec{E} in a Taylor series around the CM: $E^i = E_{\text{CM}}^i + (E_{\text{CM}})^i{}_{,j} r^j + \dots$ (for the integral above, and to quadrupole order only terms to linear order in \vec{r} are to be kept in this expansion), which, splitting $E^{i,j} = (E_{\text{CM}})^{[i,j]} r_j + (E_{\text{CM}})^{(i,j)} r_j$, we may write as

$$E^i = E_{\text{CM}}^i - \frac{1}{2}[\vec{r} \times (\nabla \times \vec{E})_{\text{CM}}]^i + (E_{\text{CM}})^{(i,j)} r_j$$

The first term is the part of \vec{E} that has a curl, and is thus the induced electric field: $\vec{E}_{\text{ind}}(r) \approx -\vec{r} \times (\nabla \times \vec{E})_{\text{CM}}/2$ (the second part may be cast as a gradient of some scalar function and thus not related with induction). Therefore, recalling the definition of $q^{\alpha\beta}$, Eq. (111) above:

$$\begin{aligned} \tau_{\text{ind}}^i &= -\frac{1}{2}(\nabla \times \vec{E}_{\text{CM}})^j \int \rho_c [r^i r_j - \delta_j^i r^2] d^3x \\ &= -\frac{1}{2}(\nabla \times \vec{E}_{\text{CM}})^j [q_j^i - \delta_j^i q^\gamma{}_\gamma] \end{aligned} \quad (121)$$

which, by virtue of relations (20), is a non-covariant form for

$$\tau_{\text{ind}}^\alpha = \frac{1}{2}\epsilon^\sigma{}_{\mu\nu} E^{[\mu\nu]} [q^\alpha{}_\sigma - \delta^\alpha{}_\sigma q^\gamma{}_\gamma] \quad (122)$$

i.e., the first term of (120). Note that, using Eqs. (I.4a) of Table I, we can also write it as

$$\tau_{\text{ind}}^\alpha = \frac{1}{2}B^\sigma{}_\beta U^\beta [q^\alpha{}_\sigma - \delta^\alpha{}_\sigma q^\gamma{}_\gamma]. \quad (123)$$

It is interesting to consider the *special case* of a spinning body with uniform mass and charge density; we may write $2\sigma I_\beta^\alpha = (q_\gamma (h^U)^\alpha{}_\beta - q_\beta^\alpha)$, where $I_{\alpha\beta}$ is the moment of inertia tensor (see e.g. [31]), which we write covariantly as:

$$I_\beta^\alpha \equiv \int_{\Sigma(\tau,U)} J^\delta [r^\gamma r_\gamma (h^U)^\alpha{}_\beta - r^\alpha r_\beta] d\Sigma_\delta, \quad (124)$$

being $J^\alpha = -T^{\alpha\beta}U_\beta$ the mass/energy current (with respect to the observer U^α comoving with the CM), and $\sigma \equiv q/2m$ is the classical gyromagnetic ratio. In this case we may re-write (122) as

$$\tau_{\text{ind}}^\alpha = -\sigma \epsilon^\sigma{}_{\mu\nu} E^{[\mu\nu]} I_\sigma^\alpha = -\sigma B^\sigma{}_\beta U^\beta I_\sigma^\alpha. \quad (125)$$

1. Rigid spinning charged body

Consider the case that the test particle is a charged, “quasi-rigid” body [22, 43], such that its charge current

density \vec{j} , as seen in the CM frame, is $j(\vec{r}) = \rho_c \vec{\Omega} \times \vec{r}$, where the vector Ω^α , spatial with respect to U^α , denotes the angular velocity of the body. Ω^α is defined as follows: let A^α be some vector (also spatial wrt U^α) with origin at the CM, and *co-rotating* with the body; then

$$D_F A^\alpha / d\tau = \Omega_\beta^\alpha A^\beta; \quad \Omega_{\alpha\beta} = \epsilon_{\beta\alpha\mu\nu} \Omega^\mu U^\nu. \quad (126)$$

From definitions (7) and (11), we have:

$$\mu^\alpha = \frac{1}{2}\epsilon^\alpha{}_{\beta\gamma\delta} U^\delta \int r^{[\beta} j^{\gamma]} U^\sigma d\Sigma_\sigma$$

which, for the rigid body, in the CM frame, becomes:

$$\mu^i = \frac{\Omega^j}{2} \int \rho_c (\delta_j^i r^2 - r_j r^i) d^3x = \frac{\Omega^j}{2} (\delta_j^i q^\gamma{}_\gamma - q^i{}_j)$$

or, covariantly,

$$\mu^\alpha = \frac{\Omega^\beta}{2} (\delta_\beta^\alpha q^\gamma{}_\gamma - q^\alpha{}_\beta) \quad (127)$$

The rate of work done on this body by the induction torque τ_{ind} , $\mathcal{P} = \tau_{\text{ind}}^\alpha \Omega_\alpha$, is, thus, from Eq. (123), or (122):

$$\tau_{\text{ind}}^\alpha \Omega_\alpha = -B^\alpha{}_\beta U^\beta \mu_\alpha = -F_{\text{EM}}^\alpha U_\alpha \quad (128)$$

That is, we obtained precisely the work \mathcal{P}_{ind} of Sec. V A, Eqs. (95)-(96). This is the result we seek: we have just proved that the work transferred to the body by Faraday's law of induction, which, to pole-dipole order, is manifest in the projection along U^α of the dipole force F_{EM}^α , and on the variation of the proper mass $dm/d\tau$, is indeed associated to an induction torque, which causes S^2 to vary as expected (since τ_{ind}^α is not orthogonal to S^α in general). This torque was known to exist from some non-relativistic treatments in the literature [65–67], dealing with the special case of *spinning spherical* charged bodies. Only that torque is *not* manifest to pole-dipole order as it involves the second moment of the charge $q_{\alpha\beta}$, which is of quadrupole order. Note as well that in order to assign a spinning particle a moment of inertia and an angular velocity, one must go beyond dipole order, as $I_{\alpha\beta}$, Eq. (124), involves the “quadrupole” of the mass $(m_Q)_{\alpha\beta}$, Eq. (149). But the rate of work it does, $\tau_{\text{ind}}^\alpha \Omega_\alpha$, by contrast, is manifest to dipole order, since, for a rigid body, $q_{\alpha\beta}$ and Ω^α combine into the magnetic dipole moment μ^α , by virtue of Eq. (127).

2. Torque on spherical charged body

In this context, and in view of a comparison with the gravitational problem, it is interesting to consider (in Minkowski spacetime) the case of an uniform, spherical charged body, whose quadrupole moments of j^α reduce to the trace of $q_{\alpha\beta}$, and we expect the total quadrupole torque on the particle τ^α to come essentially from τ_{ind}^α .

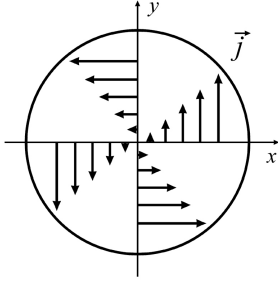


Figure 5: Current density \vec{j} for a spherical body rotating about the z axis, as seen in the CM frame. In this frame the spatial components of $\mathcal{J}^{\alpha\beta\nu}$ vanish: $\mathcal{J}^{ijk} = 0$. All the components involving z are zero: $\mathcal{J}^{ijz} = 0$, since $j^z = 0$; and $\mathcal{J}^{izj} = 0$ from the symmetry with respect to the equatorial plane. It is easy to see that: $\mathcal{J}^{xxx} = \int x^2 j^x d^3x = 0 = \mathcal{J}^{yyy}$, $\mathcal{J}^{xyy} = \mathcal{J}^{yxy} = \int xy j^y d^3x = 0 = \mathcal{J}^{yxx} = \mathcal{J}^{yxx}$.

So firstly let us explicitly compute the quadrupole moments for this type of body. It is clear that the charge quadrupole Eq. (111) is such that, in an orthonormal frame comoving with the center of mass, its time components are zero: $q^{00} = 0$, and its spatial part reduces to its trace: $q^{ij} = \delta^{ij} q^k_k / 3$.

Such tensor is covariantly written as:

$$q^{\alpha\beta} = \frac{1}{3} q^\tau_\tau (h^U)^{\alpha\beta}, \quad (129)$$

As for the tensor $\mathcal{J}^{\alpha\beta\gamma}$, its spatial components \mathcal{J}^{ijk} vanish in the CM frame, see Fig. 5. The only non-vanishing components are $\mathcal{J}^{ij0} = \int r^i r^j j^0 d^3x = q^{ij}$. Hence $\mathcal{J}^{\alpha\beta\gamma} (h^U)^\nu_\gamma = 0$, thus, by virtue of Eq. (113), $\mathcal{J}^{\alpha\beta\nu} = q^{\alpha\beta} U^\nu$, and therefore:

$$\mathcal{J}^{\alpha\beta\nu} = \frac{1}{3} q^\sigma_\sigma U^\nu (h^U)^{\alpha\beta} \quad (130)$$

Substituting (129) and (130) in (110):

$$Q^{\alpha\beta\gamma} = \frac{1}{2} q^\tau_\tau (h^U)^{\alpha[\beta} U^{\gamma]} = \frac{1}{2} q^\tau_\tau g^{\alpha[\beta} U^{\gamma]}. \quad (131)$$

Now let us compute the quadrupole torque exerted on the test body. From Eqs. (A8ii) and (A9) of Appendix A 1, together with Eq. (129), we have

$$S'^{\alpha\beta} = \frac{1}{3} q^\tau_\tau F^{[\alpha}_\sigma (h^U)^{\beta]\sigma}; \quad S'^\gamma = \frac{B^\gamma}{3} q^\tau_\tau. \quad (132)$$

Substituting (132) and (131) in Eq. (119), we obtain

$$\tau^{\sigma}_{\text{QEMcan}} = \frac{1}{3} q^\gamma_\gamma U^{[\alpha} F^{\beta]\lambda}_{;\lambda} \epsilon_{\alpha\beta}^{\sigma\delta} U_\delta = 0, \quad (133)$$

the second equality holding in vacuum (which is the problem at hand) by virtue of Maxwell's equations $F^{\alpha\beta}_{;\beta} = 4\pi j^\alpha$. This means that $\tau^{\sigma}_{\text{QEM}} = \tau'^\sigma$. Now in order to compute the torque Eq. (120), we must give a law of evolution for $q_{\alpha\beta}$. Eq. (129) ensures that the body is

spherical; we also demand $dq^\alpha_\alpha/d\tau = 0$, so that it has constant size; together these relations imply that $q_{\alpha\beta}$ is Fermi-Walker transported: $D_F q_{\alpha\beta}/d\tau = 0$, i.e., it has constant components in an orthonormal tetrad comoving with the particle's center of mass, as expected. Hence, from Eqs. (120) and (122), it follows that the quadrupole torque reduces to

$$\tau^{\sigma}_{\text{QEM}} = \tau^{\sigma}_{\text{ind}} + \frac{1}{6} \epsilon^{\sigma}_{\alpha\beta} a^\alpha E^\beta q^\gamma_\gamma; \quad \tau^{\sigma}_{\text{ind}} = -\frac{q^\gamma_\gamma}{3} \epsilon^{\sigma}_{\alpha\beta} E^{[\alpha\beta]} \quad (134)$$

i.e., up to acceleration dependent terms, $\tau^{\sigma}_{\text{QEM}}$ is the torque due to the induced electric field.

For a body with uniform charge and mass densities, it follows from Eqs. (124) and (111) that $q^\sigma_\sigma/3 = \sigma I^\sigma_\sigma/3 = \sigma I$, where $I = I_{zz} = I_{xx} = I_{yy}$ denotes the moment of inertia of the sphere with respect to any axis of rotation passing through its center. Thus we may put $\tau^{\alpha}_{\text{ind}} = -\sigma I \epsilon^{\alpha}_{\mu\nu} E^{[\mu\nu]} = -\sigma I B^\alpha_\beta U^\beta$. We shall write, in the CM frame, and in vector notation, the equation for the spin evolution (115) for such body:

$$\begin{aligned} \vec{\tau} &\equiv \frac{D\vec{S}}{d\tau} = \vec{\tau}_{\text{DEM}} + \vec{\tau}_{\text{ind}} + \frac{q^\sigma_\sigma}{6} \vec{a} \times \vec{E} \\ &= \vec{\mu} \times \vec{B} - \sigma I \frac{D\vec{B}}{d\tau} - \frac{\sigma I}{2} \vec{a} \times \vec{E}. \end{aligned} \quad (135)$$

to point out that this is the relativistic generalization of Eq. (1) of [65], or Eq. (6) of [66]; those non-relativistic results therein follow from Eq. (135) above by replacing $\tau \rightarrow t$, and neglecting the acceleration dependent term.

Let us now compute the work, $\tau^\sigma \Omega_\sigma$, done by the total torque, $\tau^\alpha = \tau^{\alpha}_{\text{DEM}} + \tau^{\alpha}_{\text{QEM}}$, on the particle. First note that, for a “quasi-rigid” body, the relation $S^\alpha = I^{\alpha\beta} \Omega_\beta$ holds [22]; which, for a uniform spherical body, cf. definition (124), becomes:

$$S^\alpha = I \Omega^\alpha. \quad (136)$$

Hence, assuming the proportionality $\mu^\alpha = \sigma S^\alpha$, it follows from Eq. (117) (with $d^\alpha = 0$, which is the problem at hand) that the work of the dipole torque is zero: $\tau^{\alpha}_{\text{DEM}} \Omega_\alpha = 0$. Thus, $\tau^\sigma \Omega_\sigma = \tau^{\alpha}_{\text{QEM}} \Omega_\alpha$. From Eqs. (127) and (129), we have

$$\mu^\alpha = \frac{1}{3} \Omega^\alpha q^\gamma_\gamma, \quad (137)$$

and therefore, from Eqs. (134) and (28),

$$\tau^\sigma \Omega_\sigma = \tau^{\sigma}_{\text{ind}} \Omega_\sigma + \frac{1}{2} \epsilon^{\sigma}_{\alpha\beta} a^\alpha E^\beta \mu_\sigma = \tau^{\sigma}_{\text{ind}} \Omega_\sigma - \frac{1}{2} P^{\alpha}_{\text{hidEM}} a_\alpha.$$

Now consider a setup for which the electromagnetic hidden momentum is zero: $P^{\alpha}_{\text{hidEM}} = 0$, as is the case of the application in Fig. 6a. In this case, $\tau^\sigma \Omega_\sigma = \tau^{\sigma}_{\text{ind}} \Omega_\sigma$. From Eqs. (115) and (136), we have also $\tau^\sigma = I D\Omega^\sigma/d\tau$; therefore:

$$I \frac{D\Omega^\sigma}{d\tau} \Omega_\sigma = \frac{1}{2} I \frac{d\Omega^2}{d\tau} = \tau^{\sigma}_{\text{ind}} \Omega_\sigma = -F^{\alpha}_{\text{EM}} U_\alpha \quad (138)$$

where in the last equality we used (128). $I\Omega^2/2$ is the body's kinetic energy of rotation about its CM, see e.g. [22, 43]; hence Eq. (138) above tells us that for this setup, the rate of variation of the body's kinetic energy of rotation equals the rate of work, *as measured in the CM frame*, done by the dipole force F_{EM}^α on the particle (that is, its projection $-F_{EM}^\alpha U_\alpha$ along the particle's worldline).

The condition $P_{\text{hidEM}}^\alpha = 0$ implies also, cf. Eq. (28), that $B^\alpha_\beta U^\beta = DB^\alpha/d\tau$; therefore

$$\tau_{\text{ind}}^\sigma = -\frac{1}{3}q^\gamma_\gamma B^\alpha_\beta U^\beta = -\frac{1}{3}q^\gamma_\gamma \frac{DB^\sigma}{d\tau}, \quad (139)$$

and we may write, using (137),

$$\frac{1}{2}I \frac{d\Omega^2}{d\tau} = \tau_{\text{ind}}^\sigma \Omega_\sigma = -\frac{DB^\alpha}{d\tau} \mu_\alpha. \quad (140)$$

Using $D\mu^\alpha/d\tau = \sigma DS^\alpha/d\tau$, together with Eqs. (116), (117), (134) and (139), we can rewrite Eq. (140) as:

$$\frac{1}{2}I \frac{d\Omega^2}{d\tau} = -\frac{d(B^\alpha \mu_\alpha)}{d\tau} - \frac{\sigma q^\gamma_\gamma}{6} \left(\frac{dB^2}{d\tau} - \epsilon^\sigma_{\alpha\beta} a^\alpha E^\beta B_\sigma \right) \quad (141)$$

which is the relativistic generalization of Eq. (10) of [66] (the acceleration dependent term, which arises from our demand that $q_{\alpha\beta}$ be Fermi-Walker transported, is absent therein). From this equation we see that, for this setup (a quasi-rigid, uniform, spherical charged body), the varying part of the mass, $-\vec{B} \cdot \vec{\mu}$, present in the dipole approximation, cf. Eq. (36), is *kinetic energy of rotation* (not potential energy, as claimed in some literature, e.g. [22, 36, 46]) — this establishes, in a relativistic covariant formulation, and in the context of Dixon's multipole approach, the claims in [64–67]. The remaining terms in Eq. (141), involving the charge quadrupole $q_{\alpha\beta}$, are not manifest in the (dipole order) mass equation (35), since to that accuracy $B^\alpha D\mu_\alpha/d\tau = 0$, by virtue of Eq. (23).

B. Gravitational torque

The equation for the spin evolution of an extended spinning body in a gravitational field is, up to quadrupole order [29, 38]

$$\frac{DS^{\kappa\lambda}}{d\tau} = 2P^{[\kappa}U^{\lambda]} + \frac{4}{3}J^{\mu\nu\rho[\kappa}R^{\lambda]}_{\rho\mu\nu} \quad (142)$$

leading to

$$\frac{D_F S^\sigma}{d\tau} = \tau_{\text{QG}}^\sigma \quad \tau_{\text{QG}}^\sigma \equiv \frac{4}{6}J^{\mu\nu\rho[\kappa}R^{\lambda]}_{\rho\mu\nu} \epsilon_{\kappa\lambda}^{\sigma\delta} U_\delta. \quad (143)$$

The general definition of the gravitational quadrupole moment $J^{\alpha\beta\gamma\delta}$, given by Eqs. (9.12), (9.11) and (9.4) of [38], involves bitensors (σ^κ , $\Theta^{\kappa\lambda\mu\nu}$ and $H_{\alpha\beta}$ in the notation of [38]). Let us briefly comment on that. Only in flat spacetime (and Lorentz coordinates) the separation vector between two points x^α and z^α is $r^\alpha \equiv x^\alpha - z^\alpha$

and the distance between them given by $\sqrt{r^\alpha r_\alpha}$. In the general case of a curved spacetime, the distance between two points is the length of the geodesic connecting them; r^α is not even (exactly) a vector, and the closest notion to a separation vector is the bitensor $-\sigma^\kappa(x, z)$, which is the vector tangent to the geodesic at \mathbf{z} whose length equals that of the geodesic connecting the two points [22, 27]. Also one must distinguish a coordinate system at \mathbf{x} from the one at \mathbf{z} , as the basis vectors change from point to point in a curved spacetime.

In the previous sections, dealing with pole-dipole approximations, we did not face this problem in defining the moments because cutting off the multipole expansion in the dipole term amounts to keep only terms linear in r ; since, *to first order*, r^α is a vector (see e.g. Eq. (7) of [86]), and $\sqrt{r^\alpha r_\alpha}$ the distance between two points, the use of bitensors is by definition redundant. We can always define the moments by the same expressions as in flat spacetime. And in an approximation up to quadrupole order, albeit not true generically, if the gravitational field is not too strong, we can still, to a good approximation (cf. [22] p. 513) skip the bitensors and define the moments by expressions identical to the ones in flat spacetime (where $\sigma^\kappa = -r^\kappa$, $\Theta^{\kappa\lambda\mu\nu} = g^{\kappa(\mu}g^{\mu\lambda)}$ and $H_\alpha^\lambda \simeq \delta_\alpha^\lambda$, see [22, 38]). For that, and following [4, 85], one sets up a locally nearly Lorentz Frame \hat{e}_α with origin at the reference worldline, *momentarily* comoving with the test particle (i.e., $\hat{e}_0 = U^\alpha$ and the spatial triad \hat{e}_i spans the instantaneous rest space of the center of mass). In such frame $r^{\hat{\alpha}} = x^{\hat{\alpha}}$ yields a good approximation to the displacement vector between a point x^α and the reference worldline $z^\alpha(\tau)$.

From Eqs. (5.29), (5.9) and (4.20) of [21], we have, for the case $F^{\alpha\beta} = 0$:

$$J^{\alpha\beta\gamma\delta} = \frac{1}{2} \left(t^{\gamma[\alpha\beta]\delta} - t^{\delta[\alpha\beta]\gamma} \right) - U^{[\alpha} p^{\beta][\gamma\delta]} - U^{[\gamma} p^{\delta][\alpha\beta]} \quad (144)$$

where the moments $t^{\alpha\beta\gamma\delta}$ and $p^{\alpha\beta\gamma}$, in the frame \hat{e}_α , are defined as:

$$t^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \equiv \int_{\Sigma(\tau, U)} r^{\hat{\alpha}} r^{\hat{\beta}} T^{\hat{\gamma}\hat{\delta}} w^{\hat{\sigma}} d\Sigma_{\hat{\sigma}}, \quad (145)$$

$$p^{\hat{\alpha}\hat{\beta}\hat{\gamma}} \equiv \int_{\Sigma(\tau, U)} r^{\hat{\alpha}} r^{\hat{\beta}} T^{\hat{\gamma}\hat{\delta}} d\Sigma_{\hat{\delta}}, \quad (146)$$

Noting that $T^{\gamma\delta} d\Sigma_\delta = -T^{\gamma\nu} U_\nu U^\delta d\Sigma_\delta = J^\gamma U^\delta d\Sigma_\delta$, where J^α denotes the mass/energy current, we may rewrite $p^{\alpha\beta\gamma}$ as:

$$p^{\hat{\alpha}\hat{\beta}\hat{\gamma}} = \int_{\Sigma(\tau, U)} r^{\hat{\alpha}} r^{\hat{\beta}} J^{\hat{\gamma}} U^\delta d\Sigma_\delta \quad (147)$$

which has thus the interpretation of the quadrupole moment of the mass current, analogous to the electromagnetic quadrupole moment of the charge current $\mathcal{J}^{\alpha\beta\gamma}$, Eq. (112). Again we take $w^\sigma \approx U^\alpha$ in (145) since, to quadrupole order, only terms up to r^2 are being kept in

the integrands. Therefore $t^{\alpha\beta\gamma\sigma}(\Upsilon^U)_\sigma^\delta = p^{\alpha\beta\gamma}U^\delta$, and we may decompose $t^{\alpha\beta\gamma\delta}$ as follows

$$t^{\alpha\beta\gamma\delta} = p^{\alpha\beta\gamma}U^\delta + p^{\alpha\beta\sigma}(h^U)_\sigma^\delta U^\gamma + t^{\alpha\beta\lambda\sigma}(h^U)_\lambda^\gamma (h^U)_\sigma^\delta. \quad (148)$$

$p^{\alpha\beta\gamma}$ may also be decomposed as

$$p^{\alpha\beta\gamma} = (m_Q)^{\alpha\beta}U^\gamma + p^{\alpha\beta\lambda}(h^U)_\lambda^\gamma$$

where

$$(m_Q)^{\hat{\alpha}\hat{\beta}} = \int_{\Sigma(\tau,U)} r^{\hat{\alpha}} r^{\hat{\beta}} J^\gamma d\Sigma_\gamma \quad (149)$$

is the mass quadrupole (or second moment of the mass, see [29, 31, 85] and Footnote 20), analogous to charge quadrupole (111).

1. Torque on spinning “spherical” body

Our goal in this section is to consider the gravitational analogue of the problem in Sec. VIA 2. Therein we considered a spherical charged body in flat spacetime, whose charge quadrupole moment was shown to reduce to its trace: $q_\beta^\alpha = q_\tau^\tau (h^U)_\beta^\alpha / 3$, and the quadrupole of the current to $\mathcal{J}^{\alpha\beta\nu} = q_\sigma^\sigma U^\nu (h^U)^{\alpha\beta} / 3$. We prescribe the analogous test body for the gravitational problem by demanding it to have an *analogous multipole structure* (i.e., its “gravitational skeleton” [35]) to its electromagnetic counterpart (rather than demanding its shape to be “spherical”, which in a general curved spacetime is not a well defined notion. A body with such multipole structure will of course be a sphere in the case of flat spacetime; and otherwise may be thought as one if the field is not too strong). As shown above, the quadrupole moment $p^{\alpha\beta\gamma}$, Eq. (147), has an analogous definition to $\mathcal{J}^{\alpha\beta\gamma}$, Eq. (112), only with J^α in the place of j^α ; hence its structure must be (analogously to Eq. (130)):

$$p^{\alpha\beta\gamma} = \frac{1}{3}(m_Q)^\tau_\tau (h^U)^{\alpha\beta} U^\gamma \quad (150)$$

In a local orthonormal tetrad \hat{e}_α such that $\hat{e}_0 = U^\alpha$ (i.e., the triad \hat{e}_i spans the rest space of the center of mass), this tensor has similar components to $\mathcal{J}^{\alpha\beta\gamma}$ as written in the CM frame (only with $J^{\hat{\alpha}}$ in the place of $j^{\hat{\alpha}}$). From Eq. (148) we have

$$t^{\alpha\beta\gamma\delta} = \frac{1}{3}(m_Q)^\tau_\tau (h^U)^{\alpha\beta} U^\gamma U^\delta + t^{\alpha\beta\lambda\sigma}(h^U)_\lambda^\gamma (h^U)_\sigma^\delta. \quad (151)$$

The second term of (151) is the quadrupole moment of $T^{\hat{i}\hat{j}}$, which has no electromagnetic analogue (the time projection of $T^{\alpha\beta}$: $-T^{\alpha\beta}U_\beta = J^\alpha$ has an electromagnetic analogue, which is the current density j^α ; the space part of $T^{\alpha\beta}$ has not). For a “quasi-rigid” spinning body, we have in the CM frame (e.g. [87]) $T^{\hat{i}\hat{j}} = \rho u_{\text{rot}}^{\hat{i}} u_{\text{rot}}^{\hat{j}} + s^{\hat{i}\hat{j}}$, where u_{rot}^α denotes the 4-velocity of the (rotating) mass

element of the body and $s^{\hat{i}\hat{j}}$ are the stresses. For non-relativistic rotation speeds $v_{\text{rot}} \ll 1$, $s^{\hat{i}\hat{j}}$ can be neglected [87, 88], and thus $T^{\hat{i}\hat{j}} \approx \rho v_{\text{rot}}^{\hat{i}} v_{\text{rot}}^{\hat{j}} \sim \rho v_{\text{rot}}^2$; therefore, in this regime, the second term of (151) is negligible compared to the first one (involving $T^{\hat{0}\hat{0}} = \rho$). It then follows

$$J^{\alpha\beta\gamma\delta} \approx -(m_Q)^\tau_\tau U^{[\alpha} g^{\beta][\gamma} U^{\delta]}, \quad (152)$$

(in agreement with Eq. (7.31) of [22]). Substituting in Eq. (143), we obtain the gravitational torque:

$$\tau_{QG}^\sigma = -\frac{1}{3}(m_Q)^\tau_\tau U^{[\alpha} R^{\beta]}_{\mu} U^\mu \epsilon_{\alpha\beta}^{\sigma\delta} U_\delta = 0$$

the second equality holding for vacuum ($R^{\mu\nu} = 0$), which (as in the electromagnetic case) is the problem at hand. Thus, no gravitational torque is exerted, up to quadrupole order, in a spinning spherical body. This means that there is no gravitational counterpart to the electromagnetic torque τ_{ind}^α of the analogous electromagnetic problem in Sec. VIA 2, that is generated (from the viewpoint of the particle’s frame) by the induced electric field. This is the result we expected from the discussion in Sec. IIB and in [6]. The electromagnetic torque (122) comes from the antisymmetric part of $E_{\alpha\beta}$; or, equivalently, from the time projection of $B_{\alpha\beta}$, cf. Eq. (123), which encode Faraday’s law of induction, cf. Eqs. (20) and (I.4a). The absence of an analogous effect in gravity is thus natural, since the gravito-electric tidal tensor $\mathbb{E}_{\alpha\beta}$ is symmetric, and the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$ is spatial, meaning that the dynamical effects which in electromagnetism are caused by the curl of the electric field \vec{E} , have no counterpart in the *physical* gravitational forces and torques.

C. Summarizing with a simple realization

The results in Secs. VIA and VIB entirely corroborate the discussion in Sec. V (and also Sec. IIE); namely the manifestation of electromagnetic induction and the absence of an analogous phenomenon in the *physical* gravitational forces and torques. In this context it is interesting to consider the analogous problems in Fig. 6: a spinning spherical charge moving in the field of a strong magnet (or another spinning charged body), and a spinning “spherical” mass moving in Kerr spacetime. Let us start by the electromagnetic case. A force F_{EM}^α , Eq. (I.1a) will be exerted on the particle, causing it to move (thereby gaining translational kinetic energy, at a rate $\mathcal{P}_{\text{trans}}$, Eq. (103)). And as it moves in an inhomogeneous magnetic field, a torque τ_{ind}^α is exerted upon it, due, from the the viewpoint of the observer comoving with the particle, to the electric field induced by the time-varying magnetic field. That torque will cause a variation of angular momentum S^α , and therefore of the angular velocity $\Omega^\alpha = S^\alpha/I$ of the particle (measured with respect to the comoving Fermi-Walker transported

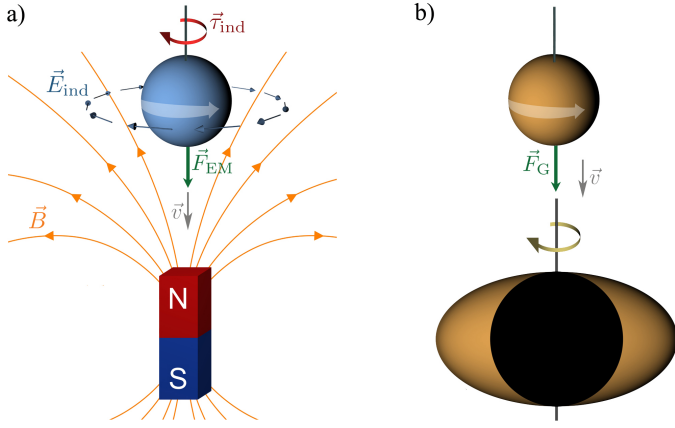


Figure 6: a) A spinning, positively charged spherical body being pulled by a strong magnet; $\vec{E}_{ind} \equiv$ electric field induced in the body's CM frame. b) A spinning spherical body falling into a Kerr Black hole. As the spinning charge moves in the inhomogeneous magnetic field \vec{B} , a torque τ_{ind}^α , Eq. (134), is exerted on it due to \vec{E}_{ind} , i.e. to the antisymmetric part $E_{[\alpha\beta]}$, or equivalently, to the time projection $B_{\alpha\beta}U^\beta$. That causes $S \equiv \sqrt{S^\alpha S_\alpha}$, and the body's angular velocity $\Omega = S/I$, [cf. Eqs. (136)-(126)], to vary. τ_{ind}^α does work at a rate $\tau_{ind}^\alpha \Omega_\alpha = \mathcal{P}_{ind}$, which exactly matches time projection of the dipole force F_{EM}^α , cf. Eq. (128). That causes the kinetic energy of rotation about the CM to decrease, manifest in a variation of proper mass $dm/d\tau$, and cancels out the gain in kinetic energy \mathcal{P}_{trans} , so that the total work transfer is zero (cf. Sec. VB). In the gravitational case no analogous induction effects occur (as expected since $\mathbb{E}_{[\alpha\beta]} = \mathbb{H}_{\alpha\beta}U^\beta = 0$): no torque is exerted on the spinning particle, its S , angular velocity Ω , and proper mass m , are constant; and there is a net work done on it by F_G^α at a rate $\mathcal{P}_{tot} = -F_G^\alpha u_\alpha$, corresponding to an increase of translational kinetic energy.

tetrad, cf. Eq. (126)). Clearly, as we see from Eqs. (135) or (134), S^2 is *not conserved* (as would be the case in a pole-dipole approximation, see Sec. II C). The variation of the magnitude Ω of the angular velocity also implies a variation of rotational kinetic energy of the particle; that is reflected in a variation of proper mass $dm/d\tau$, as shown by Eq. (128), which is precisely the one we found to exist in Secs. II E and V, encoded in the time projection of the force F_{EM}^α . As discussed in Sec. VB, if we choose a frame corresponding to the “static observers” u^α (cf. point 8 of Sec. ID, and Footnote 15) then $F_{EM}^\alpha u_\alpha = 0$, and the total energy of the particle, $E = -P_\alpha u^\alpha$, is conserved. That means that the rate of variation in kinetic translational energy \mathcal{P}_{trans} (due to the work of F_{EM}^i) is exactly canceled out by the variation of rotational kinetic energy \mathcal{P}_{ind} (due to the work of $\vec{\tau}_{ind}$), ensuring that a static magnetic field does no work.

In the gravitational case, there is also a net force F_G^α on the body, cf. Eq. (I.1b) of Table I, causing it to gain kinetic energy at a rate $\mathcal{P}_{trans} = F_G^i v_i$. But no torque is exerted on it; up to quadrupole order we have:

$$\frac{D_F S^\alpha}{d\tau} = 0; \quad S^2 = \text{constant},$$

(i.e., the spin vector of the spinning spherical mass is Fermi-Walker transported) implying also $\Omega = \text{constant}$. This is consistent with the constancy of the proper mass (and the fact that F_G^α is spatial, meaning that no work done by induction), because since there is no torque, the kinetic energy of rotation is constant. Thus also the gain in translational kinetic energy is not canceled out by a variation of rotational kinetic energy, and therefore a stationary gravitomagnetic field will do a net rate of work \mathcal{P}_{trans} on the particle.

We close this section with a few remarks. 1) The discussion herein explains the reason why, in the framework of the inertial fields from the 1+3 formalism, the analogy in Eq. (24) holds even when the fields are time-varying (while we expected it to break down due to the phenomenon of electromagnetic induction) — it is due to the fact that, by neglecting moments beyond dipole, we are in fact neglecting the *torque* τ_{ind}^α due to Faraday's induction, since the latter involves the “quadrupole” moment $q_{\alpha\beta}$ of the test particle (unlike the work $\mathcal{P}_{ind} = \tau_{ind}^\alpha \Omega_\alpha$ done by it, and the corresponding variation of proper mass, which is also due to electromagnetic induction, but already manifest in the pole-dipole approximation, as it comes in terms of the magnetic dipole moment μ^α ; this is the reason why m varies whereas S^2 is a constant in the dipole approximation). 2) The application in Fig. 6 illustrates an important aspect of the physical effect of frame dragging, and the electromagnetic analogue. For clarity, let us consider the case that initially the test bodies have no rotation. In the electromagnetic case, Fig. 6a, as the ball falls, it gains an angular velocity (relative to the FW transported tetrad) due to the torque $\vec{\tau}_{ind}$; in the gravitational case, Fig. 6b, no such rotation arises. However, from the point of view of a frame anchored to the distant stars, the angular velocity of the spinning mass indeed increases as it approaches the black hole. This can be easily seen as follows. In the linear limit, and *in the frame of the static observers*, it is well known [2, 3, 34, 57, 89, 112] that gravity becomes very similar to electromagnetism. In electromagnetism, from the point of view of static observers, the torque $\vec{\tau}_{ind}$ does not come from the induced electric field (which is zero in that frame, as the fields are static therein), but from the overall effect of the Lorentz force $dq\vec{v} \times \vec{B}$ applied to each charge element dq of the spinning ball. The gravitomagnetic “force” $\vec{v} \times \vec{H}$ leads to an entirely analogous “torque”. These, however, are pure artifacts of the reference frame, with no *local* physical existence; they can only be measured by locking the frame to the distant stars (by means of, e.g., a telescope). The ball in Fig. 6b, *if* initially $\Omega^\alpha = 0$, will never gain any rotation *relative to the local compass of inertia*; in fact, an observer sitting firmly with his tetrad on top of the ball will not detect any sign of rotation: he will not measure any Coriolis forces acting on the test particles he launches, and sees gyroscopes axes fixed. That, of course, is consistent with the constancy of S^α . In fact, *it is the frame of the static observers that rotates* relative to the compass of inertia, which is manifest in the

fact that they measure a non-zero gravitomagnetic field \vec{H} (i.e., they see test particles in geodesic motion being deflected by the Coriolis “force” $\vec{v} \times \vec{H}$, and gyroscopes “precessing”). That is, we have a gravito-electromagnetic analogy based on GEM “acceleration fields” (namely \vec{H}) in the frame anchored to the distant stars; but locally it vanishes, and this is the essence of the frame dragging effect. The electromagnetic counterparts, by contrast, and electromagnetic induction, are locally measurable quantities, which manifest themselves *covariantly* in the *physical* tidal forces and torques.

Sec. VI in brief — the torque on the spinning particle (manifestation of Faraday’s induction)

- The torque due to Faraday’s law of induction, τ_{ind}^α , comes from a coupling of $E_{[\alpha\beta]}$ to $q_{\alpha\beta}$ (the charge “quadrupole”):
 - dipole approximation ignores $q_{\alpha\beta}$, as well as the moment of inertia; that explains why τ_{ind}^α is not manifest to dipole order;
 - but the work it does, $\tau_{ind}^\alpha \Omega_\alpha$, is manifest to dipole order! It equals the projection $-F_{EM}^\alpha U_\alpha$ of the dipole force along its worldline.
- τ_{ind}^α has no gravitational analogue (consistent with $\mathbb{E}_{\alpha\beta} = 0$).
- A time-varying electromagnetic field torques a spherical charged body; the gravitational field never torques a “spherical” body; m (includes kinetic energy of rotation), S , and angular velocity relative to FW transport, are *constant*.

VII. CONCLUSION

In this work we studied the dynamics of spinning test particles in general relativity, in the framework of *exact* gravito-electromagnetic analogies. A detailed summary of the main results and realizations is given in Sec. I, so we conclude with some additional remarks.

Both the equations of motion — force and spin evolution — of a spinning particle in a gravitational field are related to their electromagnetic counterparts by *exact* analogies, valid for generic fields. And expanding the force equation explicitly as an equation for the center of mass acceleration, another exact analogy emerges, for the “hidden momentum”. The first one (the analogy based on tidal tensors for the force) was introduced by one of the authors in [6]; the second one (the analogy based on the 1+3 inertial fields for the spin equation) is also still relatively little known (to this degree of generality can be found only in [9, 10], to the authors’ knowledge); the

hidden momentum analogy, Eq. (29), is presented here, in its exact form²², for the first time. They are all herein extracted from the rigorous equations of motion for pole-dipole particles (e.g. [21, 27, 29, 35, 38, 39, 43, 90, 91]), from which they emerge *if* the Mathisson-Pirani spin condition is employed.

These analogies are useful essentially for two reasons: practical purposes, as they provide a familiar formalism to treat exactly otherwise complicated gravitational problems; and theoretical purposes, as they provide suitable tools for a transparent comparison of the gravitational and electromagnetic interactions (this is especially the case of the tidal tensor analogy, which based on physical forces, common to both theories). It is the main point of this work that there is a lot to be learned (about *both* of them) from a comparative study of the gravitational and electromagnetic interactions.

The first remark we want to make, is that it is important to realize that the existence of these *exact* analogies does *not* mean that the interactions are similar. The GEM analogies are best known in the context of linearized theory, e.g. [1–3, 92]; not that well known are the exact ones, as explained above. These however have a different status from the analogies drawn in the linearized theory, in the sense that the latter, in order to hold, require a degree of similarity between the interactions to which the exact ones mentioned above are not bound. These are *functional* analogies: the magnetic tidal tensor of electromagnetism $B_{\alpha\gamma}$ plays in the equation (Ia) for the force exerted on a magnetic dipole the same role as the gravito-magnetic tidal tensor $\mathbb{H}_{\alpha\gamma}$ in the exact Eq. (I.b) for the gravitational force exerted on a gyroscope: both forces are determined by a contraction of the spin/magnetic dipole 4-vector with a magnetic-type tidal tensor *as measured by the test particle* of 4-velocity U^α . And the functional analogy extends to Maxwell and Einstein field equations, as manifest Table I. Also, in the appropriate frame (unlike the former, which is covariant and fully general, this analogy is less general, not covariant, attached to a special reference frame, as explained in detail in [34]) the gravitomagnetic field \vec{H} plays in the “precession” of the gyroscope an analogous role to \vec{B} in the precession of a magnetic dipole, cf. Eq. (24). The analogy extends, under special conditions in stationary fields, to the equations for the geodesics, for the force on the test particle, and in the field equations, see e.g. [5, 9, 34]. But the analogies *do not imply*, even in seemingly analogous setups (for instance, the electromagnetic field of spinning charge, and the gravitational field of a spinning

²² In an approximate form it was previously obtained in [29] using the spin condition $S^{\alpha\beta} P_\beta = 0$; see Appendix C 3 b. Note also that the formal similarity of the hidden momentum term $\vec{S} \times \vec{a}$ with the electromagnetic field momentum $-\vec{\mu} \times \vec{E}$ of Eq. (D5) (opposite to the hidden momentum of electromagnetic origin) was already noticed in [5].

mass, considered in Secs. III and IV B) that these objects are similar. Firstly, the gravitomagnetic field \vec{H} and tidal field $\mathbb{H}_{\alpha\beta}$, unlike their electromagnetic counterparts, are non-linear in the potentials (except for very special cases, such as ultrastationary spacetimes, studied in e.g. Sec. IV of [6]). But that is not all; even if one goes to the weak field regime (where the non-linearities of the gravitational field can be neglected), the symmetries and the time projections of the tidal tensors continue to differ crucially. This shows how misleading can be the apparent similarity suggested by the usual linear approaches in the literature, which is detailed in Sec. III. Indeed we have seen that the electromagnetic and gravitational effects can be *completely opposite*: as shown in Sec. V for an observer comoving with the test particle, the time component of F_G^α is zero, whereas F_{EM}^α is non-zero, and corresponds to the power transferred to the dipole by induction; for static observers u^α , $F_{\text{EM}}^\alpha u_\alpha = 0$ (consequence of the fact that stationary electromagnetic fields can do no work on a magnetic dipole) whereas $F_G^\alpha u_\alpha \neq 0$, meaning that a gravitomagnetic tidal field does work on mass currents.

The exact analogies are suited instead for a comparison between the interactions: as it amounts to compare mathematical objects that play analogous dynamical roles in both theories. Such comparison unveils suggestive similarities, useful in terms of the intuition they provide. But, and especially in the case of the tidal tensor formalism, it is actually through the differences it makes transparent that it proves illuminating. The differences between the tensorial structure of the gravitational and electromagnetic tidal tensors are related to fundamental differences between the interactions, namely the phenomenon of electromagnetic induction, and the way *it manifests itself in the electromagnetic tidal forces and torques*, which has no analogue in gravity. The results in Sec. V, concerning the time components of the force, and in Sec. VI, concerning the torque exerted on the spinning particle, are manifestations of the absence of a gravitational counterpart to the antisymmetric part of $E_{\alpha\beta}$ (or, equivalently, to the projection of $B_{\alpha\beta}$ along U^α); $E_{[\alpha\beta]}$ encodes the Maxwell-Faraday law $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$; the gravito-electric tidal tensor by contrast is symmetric: $\mathbb{E}_{[\alpha\beta]} = 0$, translating in an absence of analogous induction effects in the *physical* gravitational forces and torques. And the results in Secs. IV A and IV B, showing that in a non-homogeneous gravitational field there are moving observers for which $\mathbb{H}_{\alpha\beta} = 0$, so that gyroscopes can actually move along radial or circular geodesics (in Schwarzschild and Kerr-dS spacetimes, respectively), manifest that there is no gravitational analogue to Maxwell Eq. $\nabla \times \vec{B} = \partial \vec{E}/\partial t$; as in the electromagnetic systems, due to this law (more precisely, in covariant form $2B_{[\alpha\beta]} = \star F_{\alpha\beta;\gamma} U^\gamma$), $B_{\alpha\beta}$ is non-vanishing whenever the dipole “sees” a varying field (as is the case when the particle moves in a non-homogeneous field), and therefore (except for some special orientations of $\vec{\mu}$) an electromagnetic force $F_{\text{EM}}^\alpha \neq 0$ is exerted on it.

This contrast is interesting, of course, from the theoretical point of view, as the deep connection between the two theories is still very much an open question; but it is also valuable in terms of the intuition it provides to practical applications. Acknowledging the differences in the symmetries of the tidal tensors, namely that there is no gravitational counterpart to the electromagnetic antisymmetric parts, allowed us to predict, in those conditions where the electromagnetic force Eq. (I.a) comes exclusively from $B_{[\alpha\beta]}$, that the force (I.b) would be zero in the analogous gravitational setup. In this framework we predicted (Sec. IV B 3) the existence of circular geodesics for spinning particles in Kerr-dS (which, to our knowledge, were not known in the literature²³), or that a gyroscope dropped from rest in Schwarzschild spacetime would move along a geodesic towards the singularity (Sec. IV B). Note that even the latter, which is the simplest application in this work, could be a complex problem (involving possibly complicated descriptions, and difficulties in setting up its initial conditions, see Appendix C 1) outside the tidal tensor formalism and the Mathisson-Pirani condition. We have also understood (Secs. V A and II E) the conservation of proper mass of a gyroscope from the spatial character of $\mathbb{H}^{\alpha\beta}$, as in the electromagnetic analogue the variation of proper mass arises from the work \mathcal{P}_{ind} of the induced electric field, encoded in the projection of $B_{\alpha\beta}$ along the particle’s 4-velocity U^α (or equivalently, from the anti-symmetric part of $E_{\alpha\beta}$), which has no gravitational analogue. In the same spirit we physically interpreted the spin-dependence of Hawking’s upper bound [30] for the energy released when two black holes collide: if one considers (from the point of view of static observers) a magnetic dipole falling into a strong magnet [as illustrated in Fig. 3b)], there is no net gain in energy of the dipole, as the stationary magnetic field can do no work on it. That comes from an exact cancellation (since the hidden momentum term vanishes in this case) between the loss in proper mass $\mathcal{P}_{\text{ind}} = dm/d\tau$, which consist on the rate of work done by the electric field induced in the CM frame, and the gain in translational kinetic energy $\mathcal{P}_{\text{trans}}$. Since \mathcal{P}_{ind} has no gravitational counterpart, such cancellation cannot occur, and therefore we expect a stationary gravitomagnetic tidal field to do a net work $-F_G^\alpha u_\alpha = F_G^\alpha v_\alpha$ on a “gravitomagnetic dipole” (a gyroscope); which turns out to be the case: it leads to the Hawking-Wald spin interaction energy [1]. In other words: we show that the gravitational spin interaction energy, and the spin dependence of Hawking’s upper bound for the collision energy, are justified with the fact that, *unlike its electromagnetic counterpart*, a stationary gravitomagnetic tidal field *does work*.

Its is worth noting also that, on some conditions where

²³ In the context of spinning particles in Kerr-dS spacetimes, it is worth mentioning the work [94], where another problem was addressed: equilibrium positions.

the forces differ significantly (for instance when the particle moves relative to the source), certain qualitative similarities remain, from which one can still get useful intuition to visualize gravitational effects. An example is the deviation from geodesic motion of a gyroscope (in non-radial motion) in Schwarzschild spacetime²⁴. Eqs. (I.1) show that in both cases it is the magnetic tidal tensor *as seen by the test particle* that determines the force. Hence the gyroscope deviates from geodesic motion by the same reason that a magnetic dipole suffers a force even in the coulomb field of a point charge: since it is moving in the electric field, in its “rest” frame, there is a non-vanishing magnetic tidal tensor (one must however bear in mind that in this case the two tidal tensors can be very different; and that whereas in the case of a moving dipole one has always $B_{\alpha\beta} \neq 0$, the same does not necessarily happens in the gravitational case; see Sec. IV A).

As for the other physical analogies, based on “inertial” fields — the analogy for spin precession of Sec. II C, and the hidden momentum analogy of Sec. II D, the first, though not extensively used in this work, still plays an important role, as one of the results in Sec. VI is the clarification of the reason why it holds for arbitrary fields (while one might expect it to break down for time dependent fields, due to electromagnetic induction, see Secs. II C and VI C). It is also involved in the discussion in Sec. IV B 2, of the gravitational analogue to the velocity field for which a magnetic dipole does not precess. This analogy, in the context of weak, stationary fields, is one of the best known (yet its precise mathematical meaning not often well understood; see [34]) and widely used, as it can be applied to effects such as Thomas-Precession [9, 10, 72] or the Lense-Thirring effect [3, 9, 10, 56, 83]. The hidden momentum analogy, by its turn, is used in the companion paper [25] to explain the dynamics of the famous Mathisson helical motions, and in [29] to explain the bobbings observed in numerical simulations of binary systems (though therein not using the exact analogy in Eq. (29), but an approximate one obtained by employing Tulczyjew-Dixon spin condition instead). In Appendix D 1 we interpret the formal similarity between the “inertial” $\vec{S} \times \vec{G}$ and electromagnetic $\vec{\mu} \times \vec{E}$ hidden momenta, through an analogue model.

The purely formal analogy between the Maxwell and Weyl tensors (their decompositions in electric and mag-

netic parts, and their scalar invariants) we use in Sec. IV B differs from the physical analogies discussed above in the sense that the parallelism drawn is not between objects playing analogous dynamical roles (see Sec. 4.3 of [14] and [34]) in the two theories. Yet it was useful in this work as it provided familiar, intuitive tools to solve the problem in Sec. IV B. From the structure of the scalar invariants of $F_{\alpha\beta}$ for the electromagnetic field of a spinning charge, we know that in the equatorial plane there are velocity fields U^α for which the magnetic field B^α vanishes, thus for which a magnetic the dipole does not precess. Given the formal similarity with the quadratic invariants of $R_{\alpha\beta\gamma\delta}$, we anticipated that in the analogous gravitational setup (Kerr spacetime, which asymptotically, describes the field of a spinning mass) there is a velocity field for which $\mathbb{H}_{\alpha\beta} = 0$. One must always however bear in mind the non-physical character of the analogy; as $\mathbb{H}_{\alpha\beta} = 0$ means *not* that the gyroscope does not precess (which it does, for this velocity field, as shown in [15]) but that it feels no force. (Again this exemplifies, now in the more obvious context of a non-physical, purely formal one, that the existence of an analogy does not imply any similarity; as the effects at stake are in this case actually totally opposite. This observation is crucial for the understanding of the physical meaning of the scalar invariants of the Riemann tensor, which is discussed in the companion paper [15].)

In the course of this work a clarification of a number of issues concerning the dynamics of spinning particles in general relativity was needed. Firstly the problem of the covariant, relativistic equations of motion for particles with pole-dipole gravitational and electromagnetic moments; the gravitational part is well established, but difficulties exist in the electromagnetic part, there being different versions of the equations in the literature and inconsistencies in their physical interpretation. Their clarification is the purpose of Appendix A 1. Then in Appendix B we dissect the differences between the dynamics of electric and magnetic dipoles (which may be cast as the contrast of the two dipole models: a current loop in the magnetic case, a pair of opposite monopoles in the electric case), which is reflected in the spin vector evolution, hidden momentum, proper mass (and its variation), and the force exerted on the particle (both its time and space projections). In particular, the time projections of these forces, their physical content, and relationship with the mass of the particle and the work done by the fields, is ignored in most literature, or misunderstood (e.g. [22, 36, 46]). (Failure to notice that the force is not spatial, and the proper mass not constant, has even led some authors [16, 17] to believe that a covariant treatment in the dipole scheme is not possible.) These are rigorously discussed, for the case of the magnetic dipole, in Secs. V and II E; and in Appendix B for the electric dipole. In Sec. V A we clarify the meaning of the projection of F_{EM}^α parallel to U^α , as being the rate work transferred to the dipole by Faraday’s induction, and its relation with the variation of proper mass m ; the

²⁴ Note that this application is beyond the scope of previous approaches to the Papapetrou equation in the context terms of an analogy with electromagnetism. As we have seen in Sec. III, the expression $\vec{F}_G = -\nabla(\vec{S} \cdot \vec{B}_G)$ of the usual linearized theory (e.g. [2]) is valid only when the gyroscope is at rest (in a stationary spacetime). Hence, this expression can only account for the coupling between the spin of the gyroscope and the spin of the source (that is zero for Schwarzschild black hole). The exact equation (5) of [5], derived in the framework of the Quasi-Maxwell formalism is valid only for stationary spacetimes, and when the gyroscope’s worldline is tangent to a time-like Killing vector.

later being further discussed in Secs. IIE and VIA. In particular we clarify that, contrary to usual claims in the literature, e.g. [22, 36, 46], but in agreement with the point of view in [39], the contribution $-\vec{\mu} \cdot \vec{B}$ to m is *not* potential energy (as \vec{B} does no work), but internal energy of the particle, which, for a spherical quasi-rigid spinning charged body, is shown in Sec. VIA 2 to be but kinetic energy of rotation about the center of mass [cf. Eq. (141)]. That is in agreement with the results previously obtained in the non-relativistic treatments [64–67]; as for these, our treatment in Secs. VIA 2, IIE shows how they fit in the relativistic multipole scheme (namely that the kinetic energy of rotation is encoded in m). These are aspects of the dynamics of magnetic dipoles (and comparison with the gravitational case) that are central in this work; they involve however many subtleties which, we believe, can only be fully assimilated by contrasting with the case of the electric dipole, which we do in Appendix B.

Another important clarification regarding the electromagnetic equations was made in Sec. VIA, where we studied Dixon’s [29, 38] equations to quadrupole order. The dipole torque in Eq. (22) (assuming $\vec{\mu} = \sigma \vec{S}$), only causes the magnetic dipole to precess (since $\vec{\tau} \perp \vec{S}$); its spin vector would have constant magnitude $S \equiv \sqrt{S^\alpha S_\alpha}$ accordingly, regardless of the time dependence of the field. This seems puzzling, as if one thinks about the magnetic dipole as arising from a spinning charged body, placed in a time varying magnetic field, a torque $\vec{\tau}_{\text{ind}}$ (see Fig. 6) due to the induced electric field would be exerted on it, changing its angular momentum. This torque is known to exist as it has been computed (for some special cases) in some non-relativistic treatments [65–67]. The constancy of S would also apparently be in contradiction with the varying proper mass, and our assessments in Secs. VA, IIE that that variation reflects the work done on the dipole by the induced electric field. In Sec. VIA we clarify these apparent contradictions, and how $\vec{\tau}_{\text{ind}}$ fits in Dixon’s multipole formalism, which indeed involves some subtleties: i) the torque (122) is not manifest to dipole order [where only the multipoles of first order in r are kept, cf. integrals (5), (7)] because it involves the “quadrupole” moment (111) (i.e., the second moment of the charge, of order r^2); however the work done by it, and the corresponding variation of m , are manifest in dipole order, as they can be written in terms of $\vec{\mu}$ (that is: the variation of angular momentum S caused by τ_{ind}^α is not manifest to dipole order, whereas the corresponding variation in kinetic energy of rotation is!). ii) The existence of this torque is overlooked in the literature [29, 38] concerning Dixon’s quadrupole order equations (which is down to the unclear physical significance of the quantities involved in the original derivation); in their usual form they are equations for the “canonical” angular momentum $(S_{\text{can}})^{\alpha\beta}$, Eq. (A3); *not* for the physical angular momentum $S^{\alpha\beta}$. iii) The confusion between canonical and physical angular momentum, and subsequent conflicts from the apparent absence of $\vec{\tau}_{\text{ind}}$, are common also

in the traditional treatments of classical rotors, see [65] for details. This clarification is of special relevance in the context of this work as the non-existence of an analogous gravitational torque is at the root of fundamental differences between the two interactions emphasized herein.

In Appendix C1 we compare the description given by different spin conditions, in particular the Dixon-Tulczyjew and the Mathisson-Pirani conditions, clearing up some common misconceptions in the literature (Sec. C2). As for the latter condition, on which the exact analogies in this work rely, it has been subject of skepticism due to the exotic helical solutions it admits (in addition to more intuitive non-helical one), which have been deemed unphysical in some literature [27, 36, 46, 47]; this is dissected in the companion paper [25], where these assessments are shown to be unfounded, and this spin condition to be perfectly valid. And herein (Sec. C1) we exemplify, with the applications in this work, that it can be the most suitable one.

As future direction, we plan an investigation of the gravito-electromagnetic analogies in the equations of motion for spinning particles to quadrupole and higher orders in the multipole expansion.

Appendix A: The equations of motion for spinning test particles in the literature

It is perhaps surprising that the problem of the covariant equations describing the motion of spinning particles subject to gravitational and electromagnetic fields is still not generally well understood, with different methods and derivations leading to different versions of the equations, and the relation between them not being clear. And that it is the electromagnetic (not the gravitational) field that has been posing more problems. In some works, e.g. [18–23], equations of motion been presented which are symmetric in terms of electric and magnetic dipoles. If not properly interpreted, this would lead to physically inconsistent predictions, given the different nature of the two dipole models (see discussion below and Appendix B). Some authors [16, 17] have even concluded that such covariant description is not possible.

The equations of motion for spinning pole-dipole particles are derived in rigorous and unambiguous forms in [39], for special relativity, and in [27] in the context of general relativity; rigorous derivations are also in [21, 22, 29, 38]; in this case, however, one must be aware of the subtleties involving the interpretation of the quantities showing up in the equations (namely, the tensors P^α and $S^{\alpha\beta}$ therein). Herein we will explain how equations (8)-(9) compare to the literature.

Eqs. (8)-(9) correspond to equations (6.31)-(6.32) of [27] (cf. also (3.1)-(3.2) of [36]), with the following simplifications in the definitions of P^α and $S^{\alpha\beta}$ ($\equiv J^{\alpha\beta}$ in [27]): 1) the bitensor $-\sigma^{\alpha\beta}$, which generalizes the concept of separation vector of special relativity $r^\alpha \equiv x^\alpha - z^\alpha$ (see also e.g. Appendix A.1 in [22]), is approximated by

r^α . Clearly this is not exactly a vector in general relativity; but it is a vector to first order (for an explicit demonstration, see e.g. Eq. (7) of [86]), whose magnitude yields also a first order estimate of the distance between two points. And since we are cutting off the expansion in the dipole term, which is linear in x^α , indeed, to this degree of accuracy, $\sigma^\alpha = -r^\alpha$. For the same reasons, the bitensor of geodesic displacement \bar{g}^κ_α becomes $\simeq \delta^\kappa_\alpha$ to this degree of accuracy. In this way, (4) and (5), agree with the definitions for P^α and $S^{\alpha\beta}$ given in e.g. [1, 81]. 2) The vector w^γ in the definition of $Q_{\alpha\beta}$ (more precisely in the the moments $j^{\alpha\beta}$ in [21, 27, 37]), which is a vector such that displacement of every point by $w^\gamma d\tau$ maps $\Sigma(\tau)$ into $\Sigma(\tau + d\tau)$ is, to pole-dipole order, to be taken as $w^\gamma \simeq U^\gamma$. The justification is more easily seen considering the case of flat spacetime²⁵ [21, 95], where we have, for Σ orthogonal to U^α :

$$w^\gamma = U^\gamma \left(1 - \frac{r_\alpha}{U_\beta U^\beta} \frac{DU^\alpha}{d\tau} \right) = U^\gamma (1 + r_\alpha a^\alpha) . \quad (\text{A1})$$

This leads to a quadrupole-type contribution to $j^{\alpha\beta}$ [21] if $a^\alpha \neq 0$:

$$\begin{aligned} j^{\alpha\beta} &\equiv \int_{\Sigma(\tau, U)} r^\alpha j^\beta w^\gamma d\Sigma_\gamma \\ &= \int_{\Sigma(\tau, U)} r^\alpha j^\beta U^\gamma d\Sigma_\gamma - a_\sigma \int_{\Sigma(s, u)} r^\sigma r^\alpha j^\beta U^\gamma d\Sigma_\gamma , \end{aligned}$$

which is negligible in this scheme. 3) The moments P^α , $S^{\alpha\beta}$ and $Q^{\alpha\beta}$ are defined relative to an hypersurface of integration $\Sigma(\tau, U)$ which is normal to U^α , as done in [21], whereas in e.g. [22, 27] hypersurfaces $\Sigma(\tau, P)$ orthogonal to P^α are used. That does not change the shape of the equations to dipole order though; in the purely gravitational case ($F^{\alpha\beta} = 0$) we know it²⁶ from the conservation equations $T^{\alpha\beta}_{;\beta} = 0$. In the electromagnetic case that can be checked comparing with [21] (identifying the appropriate quantities, see Sec. A 1 below), or with independent derivation in [39], to pole-dipole order the form of the equations remains the same.

1. Dixon's equations

In later works by Dixon [21, 22, 37, 38] these equations are presented in a different form, e.g. Eqs. (1.33)-(1.34)

of [38], which are symmetric with respect to the electric and magnetic dipoles. Taking into account the different signature and conventions, they read:

$$\frac{D(P_{\text{Dix}})^\alpha}{d\tau} = q F^{\alpha\beta} U_\beta + \frac{1}{2} F^{\mu\nu;\alpha} Q_{\mu\nu} - \frac{1}{2} R^\alpha_{\beta\mu\nu} S^{\mu\nu} , \quad (\text{A2})$$

$$\frac{D(S_{\text{can}})^{\alpha\beta}}{d\tau} = 2(P_{\text{Dix}})^{[\alpha} U^{\beta]} + 2Q^{\theta[\beta} F^{\alpha]}_{\theta} . \quad (\text{A3})$$

It must be noted that $(P_{\text{Dix}})^\alpha$ and $(S_{\text{can}})^{\alpha\beta}$ (P^α , $S^{\alpha\beta}$ in the notation of [21, 22, 37, 38]) are *not* the physical momentum and angular momentum given by Eqs. (4) and (5) above, but instead contain additional electromagnetic terms, cf. definitions (5.1) and (5.2) of [22]. As shown in [84], $(P_{\text{Dix}})^\alpha + q A^\alpha \equiv (P_{\text{can}})^\alpha$ and $(S_{\text{can}})^{\alpha\beta}$ have the interpretation of “canonical momenta” associated to the Lagrangian of the system²⁷. $(P_{\text{can}})^\alpha$ is the quantity conserved in collisions [84], and its time part is the scalar conserved in time-constant fields, cf. Eq. (B8) below (see also Eq. (5.3) of [22], Eq. (29) of [29]). $(S_{\text{can}})^{\alpha\beta}$, as explained below, generalizes the canonical angular momentum of some non-relativistic treatments [65–67]. To pole-dipole order we may skip the bitensors in these definitions, to obtain:

$$(P_{\text{Dix}})^\alpha = P^\alpha + P'^\alpha , \quad P'^\alpha \equiv \int_{\Sigma(z, U)} \Psi^\alpha j^\beta d\Sigma_\beta , \quad (\text{A4})$$

$$(S_{\text{can}})^{\alpha\beta} = S^{\alpha\beta} + S'^{\alpha\beta} , \quad S'^{\alpha\beta} \equiv 2 \int_{\Sigma(z, U)} r^{[\alpha} \Phi^{\beta]} j^\gamma d\Sigma_\gamma , \quad (\text{A5})$$

with

$$\Psi^\alpha \equiv - \int_0^1 F^\alpha_\beta(\mathbf{z} + u\mathbf{r}) r^\beta du , \quad (\text{A6})$$

$$\Phi^\beta \equiv - \int_0^1 u F^\alpha_\beta(\mathbf{z} + u\mathbf{r}) r^\beta du . \quad (\text{A7})$$

Note that in the case flat spacetime, these expressions are *exact* [21]. The lowest order approximation to these integrals is to take only the zeroth order term in the expansion of $F^{\alpha\beta}$ around z , i.e., to take $F^{\alpha\beta} \approx \text{constant along the particle's size}$; this is the problem at hand, as higher terms in the expansion of $F^{\alpha\beta}$ would lead to contributions of higher multipole moments to P'^α and $S'^{\alpha\beta}$. Thus we obtain:

$$P'^\alpha = -F^\alpha_\gamma d^\gamma \quad (i) , \quad S'^{\alpha\beta} = F^{[\alpha}_\sigma q^{\beta]\sigma} \quad (ii) , \quad (\text{A8})$$

where $q^{\alpha\beta}$ is “the charge quadrupole moment”, defined by (3.9) of [21]. As such $S'^{\alpha\beta}$ is negligible to pole-dipole

²⁵ It suffices for this purpose to work in flat spacetime; a generalization of w^α to curved spacetime only amounts to general relativistic corrections to something which is already negligible in Special Relativity.

²⁶ In the pole-dipole approximation only terms linear in r are kept; since to first order the spacetime can always be taken as flat, then by the same flat spacetime arguments (see e.g. [31]) one can show that $T^{\alpha\beta}_{;\beta} = 0$ implies that the integrals (4)-(5), defined at a point $z(\tau)$, over an hypersurface $\Sigma(\tau, u)$, are independent of u^α .

²⁷ We thank A. Harte for discussions on this point.

order, but it is of crucial importance in Sec. VI, where terms up to quadrupole order are kept. Substituting P'^{α} in equations (A2)-(A3) (and taking $S^{\alpha\beta} \simeq (S_{\text{can}})^{\alpha\beta}$), we immediately obtain equations (8)-(9).

We note also that $(S_{\text{can}})^{\alpha\beta}$ is a generalization of the concept of canonical angular momentum which arose in classical (non-covariant) treatments on electromagnetism [65–67]. In [67], a canonical angular momentum vector, Eq. (31) therein, is obtained differentiating $\partial\mathcal{L}/\partial\vec{\omega}$ ($\mathcal{L} \equiv$ Lagrangian of the system, $\vec{\omega} \equiv$ angular velocity of the body). Such 3-vector is but a non-covariant form for the spatial²⁸ vector $(S_{\text{can}})^{\gamma} \equiv \epsilon^{\gamma}_{\mu\alpha\beta} (S_{\text{can}})^{\alpha\beta} U^{\mu}/2$, as can be easily shown. $(S_{\text{can}})^{\gamma} = S^{\gamma} + S'^{\gamma}$, with

$$\begin{aligned} S'^{\gamma} &\equiv \frac{1}{2} \epsilon^{\gamma}_{\mu\alpha\beta} S'^{\alpha\beta} U^{\mu} = \frac{1}{2} [B^{\gamma} q_{\alpha}^{\alpha} - B^{\alpha} q_{\alpha}^{\gamma}] \\ &= \frac{B^{\alpha}}{2} [\delta^{\gamma}_{\alpha} q_{\sigma}^{\sigma} - q_{\alpha}^{\gamma}] , \end{aligned} \quad (\text{A9})$$

where we used the orthogonality conditions $q^{\alpha\beta} U_{\alpha} = q^{\alpha\beta} U_{\beta} = 0$. If the body has uniform mass and energy density, we can write $S'^{\gamma} = (q/2m) B^{\alpha} I_{\alpha}^{\gamma}$, where $I^{\alpha\beta}$ is the moment of inertia defined in Eq. (124). In this way, in the center of mass frame ($U^i = 0$), $(S_{\text{can}})^{\gamma} = (0, \vec{S}_{\text{can}})$ which reduces to Eq. (31) of [67].

The distinction between P_{Dix}^{α} , $S_{\text{can}}^{\alpha\beta}$ and the physical momenta P^{α} , $S^{\alpha\beta}$ is crucial; overlooking it in Eqs. (A2)-(A3) will then lead one to believe that electric and magnetic dipoles are dynamically similar, since those Eqs. are symmetric with respect to them. That, given the different nature of the two dipole models — the current loop (magnetic dipole) and the two opposite charges (electric dipole) — leads to physically inconsistent predictions: i) the electric dipole would have a hidden momentum $\vec{B} \times \vec{d}$, just like a magnetic dipole has a hidden momentum $\vec{\mu} \times \vec{E}$; this would violate the conservation equations, because in the electric dipole model the integral of the electromagnetic pointing vector vanishes. ii) a static electric field would do no work on a moving electric dipole, nor in a rotating one, which we know to be false; iii) the force on an electric dipole would have a time projection in the CM frame exactly analogous to its magnetic counterpart (whose the time projection is the rate of work done on the current loop by the induced electric field), which would seem to imply that the electric dipole is a current loop of magnetic monopoles. iv) also the spatial part of the force would not be consistent with the results known from previous works on classical electromagnetism. The properties of the electric dipoles, with an account of these

problems and a comparison with the magnetic dipole are presented in detail in the next section.

Appendix B: The electric dipole

In order to better understand some key issues in this work: the physical meaning of the time projection of the force on a magnetic dipole, the variation of its proper mass, the work done on it by the external fields and their relationship with Faraday's law of induction, as well as the hidden momentum, it is useful to contrast with the case of an electric dipole.

It is clear from Eqs. (8)-(9) that both the force and the equation describing the evolution of $S^{\alpha\beta}$ are different for electric and magnetic dipoles. These differences arise from the very intrinsic differences between the two dipole models: a system of two close monopoles of opposite charges, which is the case of the electric dipole, and the current loop model, which is the case of the magnetic dipole. For an electric dipole ($\mu^{\alpha\beta} = 0$, $q = 0$) in flat spacetime Eqs. (8)-(9) read:

$$\frac{DP^{\alpha}}{d\tau} \equiv F_{\text{el}}^{\alpha} = E^{\alpha}_{\beta} d^{\beta} + F^{\alpha}_{\beta} \frac{Dd^{\beta}}{d\tau} \quad (\text{B1})$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha} U^{\beta]} + 2d^{[\alpha} F^{\beta]}_{\gamma} U^{\gamma} , \quad (\text{B2})$$

where $E_{\alpha\beta} \equiv F_{\alpha\gamma;\beta} U^{\gamma}$ is the electric tidal tensor [6]. Firstly we note that, unlike its magnetic counterpart Eq. (I.1a) of Table I, the force on an electric dipole is not generically given by a contraction of a rank 2 tidal tensor with the dipole moment vector (only if $Dd^{\alpha}/d\tau = 0$). Indeed, the force on an electric dipole is not entirely a tidal effect: there is the extra term $F^{\alpha}_{\beta} Dd^{\beta}/d\tau$ (overlooked in most literature) which does not involve derivatives of the electromagnetic field. This term is physically interpreted as follows. From Eq. (3.23a) of [21] we have:

$$\frac{Dd^{\gamma}}{d\tau} = \mathcal{J}^{\gamma} - U^{\gamma} q$$

where q is the total charge and Dixon's moment \mathcal{J}^{α} , defined as $\mathcal{J}^{\alpha} = \int_{\Sigma(U,s)} j^{\alpha} w^{\gamma} d\Sigma_{\gamma}$, is the integral of the current density j^{α} in the 3-space orthogonal to U^{α} . Thus, for the electric dipole ($q = 0$), the extra term reduces to $F^{\alpha}_{\gamma} \mathcal{J}^{\gamma}$ and Eq. (B1) may be re-written as:

$$F_{\text{el}}^{\alpha} = E^{\alpha}_{\beta} d^{\beta} + F^{\alpha}_{\beta} \mathcal{J}^{\beta} . \quad (\text{B3})$$

Hence, the term $F^{\alpha}_{\gamma} \mathcal{J}^{\gamma}$ has a straightforward physical interpretation: if the dipole vector d^{α} varies along τ (e.g., if the dipole rotates) it generates a net electric current in the CM frame; therefore, a magnetic force $F^{\alpha}_{\gamma} \mathcal{J}^{\gamma}$ is exerted on the dipole, in addition to the tidal term $E^{\alpha}_{\beta} d^{\beta}$. This magnetic force is not a tidal effect, it is simply the coupling of $F^{\alpha\beta}$ to the current j^{α} .

²⁸ Note that definition of $(S_{\text{can}})^{\gamma}$ is not a dualization of $(S_{\text{can}})^{\alpha\beta}$, as neither $(S_{\text{can}})^{\alpha\beta}$ or $S'^{\alpha\beta}$ are spatial tensors if we take the Pirani condition $S^{\alpha\beta} U_{\beta} = 0$ (not $(S_{\text{can}})^{\alpha\beta} U_{\beta} = 0$ or $(S_{\text{can}})^{\alpha\beta} P_{\beta} = 0$, as done in [21] or [22], respectively). Hence $S'^{\alpha\beta}$ and $(S_{\text{can}})^{\gamma}$ do not contain the same information as the corresponding two-forms (only their spatial part).

Secondly we note that in the tidal term $E^{\alpha\beta}d_\beta$ the indices of the tidal tensor are reversed compared to the force on the magnetic dipole Eq. (I.1a). Since the tidal tensors $E_{\alpha\beta}$, $B_{\alpha\beta}$ are spatial in the first, but *not* in the second index, this signals a fundamental difference, which is discussed in Sec. B2 below: *by contrast* with $F_{\text{EM}}^\alpha = B^{\beta\alpha}\mu_\beta$, the tidal term $E^{\alpha\beta}d_\beta$ has zero time projection in the CM Frame, which means that, as measured *in that frame*, it does no work on the dipole.

1. No hidden momentum for electric dipole

Unlike the current loop, a two-monopole type of dipole cannot store hidden momentum of electromagnetic origin, see e.g. [59]. The expression for the momentum of an electric dipole is obtained contracting Eq. (B2) with U_β , leading to ($U^\alpha d_\alpha = 0$) :

$$P^\alpha = mU^\alpha - \frac{DS^{\alpha\beta}}{d\tau}U_\beta = P_{\text{kin}}^\alpha + P_{\text{hid}}^\alpha \quad (\text{B4})$$

showing that the only hidden momentum present is the pure gauge one P_{hid}^α arising from the spin condition (which exists regardless of the electromagnetic multipole structure of the particle). That is also to be expected from conservation arguments, as unlike its magnetic counterpart, the electric dipole does not generate electromagnetic momentum (cross momentum P_\times^α , see Appendix D) when placed in an electromagnetic field; the integral of the electromagnetic pointing vector vanishes [96]. Now consider, for simplicity, a *stationary* configuration; in this case the conservation Eqs. $(T_{\text{tot}})^{\alpha\beta}_{;\beta} = 0$ imply $P_{\text{tot}}^\alpha = 0$; if the dipole was to have any hidden momentum, it would *not* thus be canceled out by the electromagnetic field momentum, and violate the conservation equations.

This gives a strong hint on how crucial it is to distinguish between the physical momentum P^α and the canonical momentum $(P_{\text{Dix}})^\alpha = P^\alpha + P'^\alpha$ in Eqs. (A2)-(A3). These equations are symmetric in terms of electric and magnetic dipoles; hence, as can be checked contracting (A3) with U_α , or directly from (A8), $(P_{\text{Dix}})^\alpha$ includes a term $\epsilon_{\theta\mu\sigma}^\alpha d^\theta B^\mu U^\sigma$ analogous to the hidden momentum $P_{\text{hidEM}}^\alpha = \epsilon_{\theta\lambda\tau}^\alpha \mu^\theta E^\lambda U^\tau$ of the magnetic dipole:

$$P'^\alpha = P'^\alpha U_\alpha U^\alpha - \epsilon_{\theta\mu\sigma}^\alpha d^\theta B^\mu U^\sigma$$

Thus, if one overlooks the fact that $(P_{\text{Dix}})^\alpha$ is not the physical momentum, one is led to believe that the electric dipole has a hidden momentum just like a magnetic dipole; which not only would make no sense for a two-monopole model of dipole, as it would violate the conservation equations.

2. Proper mass and time projection of the force in C.M. frame

We have seen in Sec. V A that the force on a magnetic dipole is not a spatial vector; it has a time projection in the particle's proper (CM) frame, which we showed to be the power transferred to it by Faraday's induction; and which is manifest in a variation of the dipole's proper mass, cf. also Sec. II E. It is thus important to contrast with the case of the electric dipole, where such induction phenomena cannot occur, due to the different nature of the dipole model.

Contracting (B1) with U^α one obtains:

$$F_{\text{el}}^\alpha U_\alpha = -E_\gamma \frac{Dd^\gamma}{d\tau} = -E_\gamma \mathcal{J}^\gamma \quad (\text{B5})$$

where $E^\alpha \equiv F^{\alpha\beta}U_\beta$ is the electric field as measured by the test particle. Hence, also in this case the time projection is generically non-vanishing. However, it is very different from the time projection of the force on a magnetic dipole, Eq. (96); the former has nothing to do with electromagnetic induction, as it arises from the variation of the dipole vector d^α along the test particle's worldline, and *not* from the variation of the field as in the magnetic case. Indeed, since the electric dipole has no hidden momentum of electromagnetic origin and thus $P^\alpha a_\alpha = 0 \Rightarrow dm/d\tau = -U_\alpha DP^\alpha/d\tau$, we also have

$$\frac{dm}{d\tau} = -F_{\text{el}}^\alpha U_\alpha = E_\gamma \frac{Dd^\gamma}{d\tau} \quad (\text{B6})$$

(compare with Eq. (34)). In particular, we note that assuming the dipole vectors d^α to be constant along the particle's worldline, we have $F_{\text{el}}^\alpha U_\alpha = dm/d\tau = 0$, which contrasts with the situation for a magnetic dipole, where $dm/d\tau$ is zero only if $DB^\alpha/d\tau = 0$, and *not* $D\mu^\alpha/d\tau = 0$.

The result (B5) makes sense: \mathcal{J}^γ is the total current as measured in the dipole's frame. When it is non-vanishing (for instance, due to a rotation of the dipole) a non-vanishing work, in this frame, is done on the dipole by the electric field. Note that the density of Lorentz force on a arbitrary current is $f^\alpha = F^{\alpha\beta}j_\beta$ (see e.g. [70]); if the current originates from a moving charge, the force is spatial with respect to the charge's velocity U^α , since $j^\alpha = \rho U^\alpha$ (i.e., U^α is parallel to the current) and thus $f^\alpha U_\alpha = 0$. But that is not the case when the current arises from rotation or oscillation of a dipole; in this case j^α is not parallel to U^α (and the 3-current j is non-zero in the test particle's frame) and f^α is not spatial with respect to U^α .

3. Time component as measured by static observers and work done by the electric field

As discussed in Sec. V, the projection of the force along a congruence of time-like vectors u^α yields the rate of work done on the test particle as measured by the

observers $\mathcal{O}(u)$ of 4-velocity u^α . And we have seen in Sec. VB, for the case of a *magnetic* dipole, that if a congruence u^α exists along which $F_{\alpha\beta}$ is covariantly constant $F_{\alpha\beta;\gamma}u^\gamma = 0$ (which we dub *static* observers, cf. Sec. VB), the projection of the force along it $-F_{\text{EM}}^\alpha u_\alpha$ (i.e., the time component as measured in the $u^i = 0$ frame) vanishes, meaning that no work is done on the dipole. And that was understood through an exact cancellation between the variation of translational kinetic energy of its center of mass, and the variation of its internal energy (the latter being, from the point of view of the CM frame, the work done on the dipole by Faraday's induction, cf. Sec. VA).

In the case of the electric dipole, since it is not a current loop, the situation is different, as expected. The projection of the force along a congruence u^α for which $F_{\alpha\beta;\gamma}u^\gamma = 0$ is *not* zero, by contrast with its magnetic counterpart. Let again U^α be the 4-velocity of the test particle, related to u^α by Eq. (90); we have:

$$\mathcal{P} = -F_{\text{el}}^\alpha u_\alpha = \gamma(E^u)_{\beta\gamma} d^\gamma v^\beta + (E^u)_\alpha \frac{Dd^\alpha}{d\tau} \quad (\text{B7})$$

where $(E^u)^\alpha \equiv F^{\alpha\beta}u_\beta$ and $(E^u)_{\beta\gamma} \equiv F_{\beta\mu;\gamma}u^\mu$ are, respectively, the electric field and electric tidal tensor measured by the observers $\mathcal{O}(u)$.

Thus generically $-F_{\text{el}}^\alpha u_\alpha \neq 0$. That is the field *does work* on the dipole, as measured by $\mathcal{O}(u)$. The first term is a quite natural result: in a inhomogeneous electric field ($(E^u)_{\alpha\beta} \neq 0$), a net force is in general exerted on a electric dipole; if it is allowed to move ($v^\alpha \neq 0$) that force does work, which (if no other forces are present) translates into a variation of translational kinetic energy. The second term contributes when $Dd^\alpha/d\tau \neq 0$, and is non-zero even if the fields are uniform. Again this result is familiar: it is the work done by the electric field when the dipole rotates or oscillates.

4. Conserved quantities, proper mass and work done by the fields

In order to better elucidate the relationship between the work done by the fields and the variation of proper mass, as well as the role of electromagnetic induction, we will compare, in a *static* electromagnetic field, three test particles: a point monopole charge, an electric dipole and a magnetic dipole. Let u^α be the 4-velocity of the *inertial* frame relative to which the fields are static. Then u^α preserves the electromagnetic field: $\mathcal{L}_u F^{\alpha\beta} = 0$, and, therefore, from the constancy of expressions (5.3) of [22], or (29) of [29], we have that:

$$P_{\text{Dix}}^\alpha u_\alpha + qA^\alpha u_\alpha = P^\alpha u_\alpha + (E^u)^\alpha d_\alpha - q\phi = \text{constant} \quad (\text{B8})$$

where $\phi \equiv -A^\alpha u_\alpha$ denotes the electric potential measured by the static observers u^α . Using Eqs. (26)-(27), it is useful to re-write (B8) as

$$m + T + V + E_{\text{hid}} = \text{constant} \quad (\text{B9})$$

where $V = -(E^u)^\alpha d_\alpha + q\phi$ is the potential energy of the particle under the action of the field, $T \equiv (\gamma - 1)m$ is the kinetic energy associated to the translation of the center of mass, and $E_{\text{hid}} = -P_{\text{hid}}^\alpha u_\alpha$ the “hidden energy” (i.e., the time component of the hidden momentum in the chosen frame), see Sec. VB. $\gamma \equiv -U^\alpha u_\alpha$, and U^α is the particle's 4-velocity, cf. Eq. (90). In this section, for simplicity, we will ignore the inertial hidden momentum P_{hidI}^β , as it is of little relevance compared to the other terms, and the motion effects induced by it pure gauge and confined to the disk of centroids, see [25, 40] (a region within the convex hull of the body's worldtube, typically much smaller than it). Thus, we take herein $P_{\text{hid}}^\beta \approx P_{\text{hidEM}}^\beta$.

Point monopole charge ($d^\alpha = P_{\text{hid}}^\alpha = 0$). — In this case, condition (B9) means $m + T + q\phi = \text{constant}$. There is no exchange of energy with the proper mass of the particle, which is a constant:

$$\frac{dm}{d\tau} = -\frac{DP_\alpha}{d\tau}U^\alpha = -qF_{\alpha\beta}U^\alpha U^\beta = 0.$$

This only tells us that, in a time independent electromagnetic field the familiar notion of “total mechanical energy” of the particle — kinetic energy T , plus the potential energy $V = q\phi$ — is a constant of the motion. Therefore, in the absence of other external forces, every gain in kinetic energy must come from the electric potential energy V , hence there is no doubt that the electric field $\vec{E}(u) = -\nabla\phi$ is doing work. The rate of work \mathcal{P} is given by the time projection of the Lorentz force $F_L^\alpha = qF^{\alpha\beta}U_\beta$:

$$\mathcal{P} = -F_L^\alpha u_\alpha = q\gamma(E^u)^\alpha v_\alpha = -\frac{dV}{d\tau} = F_L^\alpha v_\alpha.$$

Electric dipole ($q = P_{\text{hid}}^\alpha = 0$). — Condition (B9) means in this case $m + T - (E^u)^\alpha d_\alpha = \text{constant}$; from Eq. (B6) the proper mass m is not constant; this means that energy is exchanged between the three forms: potential energy $V = -(E^u)^\alpha d_\alpha$, translational kinetic energy T , and internal energy m . Two special sub-cases are particularly enlightening:

1. the case where the dipole vector is covariantly constant $Dd^\alpha/d\tau = 0$, implying $dm/d\tau = 0$; in this case the energy exchange is similar to the monopole charge: every gain in translational kinetic energy comes from the potential energy V . It is clear that the electric tidal field is doing work, at a rate (cf. Eq. (B7)):

$$\mathcal{P} = -F_{\text{el}}^\alpha u_\alpha = \gamma(E^u)_{\beta\gamma} d^\gamma v^\beta = -\frac{dV}{d\tau} = F_{\text{el}}^\alpha v_\alpha.$$

2. The case where the dipole's CM is at rest ($\vec{v} = 0$, $\tau = t$), i.e., $T = 0$; in this case, $m - \vec{E} \cdot \vec{d} = \text{constant}$ (note that $\vec{E}(u) = \vec{E}$ if $\vec{v} = 0$), and the energy exchange occurs between the potential energy $V =$

$-\vec{E} \cdot \vec{d}$ and proper mass m (which includes rotational kinetic energy of the particle). A familiar case is the quasi-rigid dipole ($d^\alpha d_\alpha = \text{constant}$) which is allowed to rotate ($D\vec{d}/d\tau = \vec{\Omega} \times \vec{d}$). In this case, cf. Eq. (B7), the electric field does work at a rate:

$$\mathcal{P} = -F_{\text{el}}^\alpha u_\alpha = (\vec{\Omega} \times \vec{d}) \cdot \vec{E} = \frac{dm}{dt} = -\frac{dV}{dt}$$

This is simply the rate of work $\mathcal{P} = \vec{\tau} \cdot \vec{\Omega}$ done by the torque $\vec{\tau} = \vec{d} \times \vec{E}$ which is exerted on the dipole by virtue of Eq. (B2); that can be seen using the vector identity $(\vec{\Omega} \times \vec{d}) \cdot \vec{E} = (\vec{d} \times \vec{E}) \cdot \vec{\Omega}$. The torque $\vec{\tau}$ causes the dipole to rotate and gain kinetic energy at a rate \mathcal{P} . Again we see how important it is not to confuse P_{Dix}^α , $S_{\text{can}}^{\alpha\beta}$ in Eqs. (A2)-(A3) with the physical momenta. Overlooking the distinction would lead to the conclusion that, just like for a magnetic dipole, the electromagnetic field does no work on a moving electric dipole, nor in a rotating one — that would be true for a magnetic dipole only, where, for $\vec{\mu} = \sigma \vec{S}$, the torque $\vec{\tau} = D\vec{S}/d\tau = \vec{\mu} \times \vec{B}$, cf. Eq. (24), is orthogonal to \vec{S} , and therefore only causes the dipole to precess, implying no change in S^2 or rotational kinetic energy.

Magnetic dipole ($q = d^\alpha = 0$). — Condition (B9) means in this case $m + T + E_{\text{hid}} = \text{constant}$; if we take $\vec{\mu} = \sigma \vec{S}$, from Eq. (36) we have $m = m_0 - \mu^\alpha B_\alpha$, and thus the condition becomes $T - \mu^\alpha B_\alpha + E_{\text{hid}} = \text{constant}$. The energy exchange is between translational kinetic energy, proper mass and hidden energy. There is no potential energy involved (cf. [64–67]), which is consistent with the fact that the static field does no work on the magnetic dipole:

$$\mathcal{P} = -F_{\text{EM}}^\alpha u_\alpha = 0,$$

cf. Eq. (100). A case of interest in the context of this work is the one depicted in Fig. 3b, a magnetic dipole falling into a magnet along the field’s axis of symmetry. In this case $P_{\text{hid}}^\alpha = E_{\text{hid}} = 0$; implying $T - \mu^\alpha B_\alpha = \text{constant}$. The energy exchange is only between translational kinetic energy and proper mass; every gain in the former must come at the expense of latter (which, if the magnetic dipole is a spinning particle, is but *rotational kinetic energy*, cf. Sec. VIA 2 and [64–67]). Note that there is a force driving the dipole into motion; but what the field does is to interconvert translational kinetic energy into rotational kinetic energy [65, 66] (or other forms of internal energy, reflected in the proper mass of the dipole). The variation of proper mass is the one that, when analyzed from the CM frame of a dipole moving in an inhomogeneous field, is the work transferred to the dipole by the electric field induced by the varying (along the test particle’s worldline) magnetic field, see Fig. 3a.

Appendix C: Comparison of the Mathisson-Pirani and the Tulczyjew-Dixon spin conditions

As is clear from the discussion in Sec. II A, the two conditions are different, but equivalent, ways of describing the motion of a spinning body; they differ in the choice of a representative point in the body; hence we can say they are a matter of gauge choice. A supplementary condition is needed to close the equations of motion (8)-(9); the condition $S^{\alpha\beta} u_\beta = 0$ for some unit time-like vector u^α ensures that we are taking as reference worldline the center of mass as measured by some observer of 4-velocity u^α . Whether u^α is itself the 4-velocity of the center of mass (Pirani condition), or it is parallel to P^α (Dixon condition), or it corresponds to the static observers in Schwarzschild spacetime (Papapetrou-Corinaldesi condition), or any other type of observer (there is an infinite number of possibilities), the option should be based on convenience.

For a free particle in flat spacetime, clearly the Dixon condition provides the simplest description for the center of mass motion, which is straightline motion, as a non-degenerate solution. The Pirani condition includes in this case Dixon’s solution as a special case, but also the helical solutions which are more complicated descriptions.

However, in the presence of gravitational and electromagnetic fields, Dixon’s solution, in general, no longer coincides with any of Pirani’s; and it turns out that the latter may actually provide the simplest and more intuitive description. This is actually *always* always the case for the equation of spin evolution: in the absence of electromagnetic field (or other external torques) S^α is Fermi-Walker transported if $S^{\alpha\beta} U_\beta = 0$ holds; i.e., the gyroscope’s axis is fixed relative to a Fermi-Walker transported frame. This is the most natural description: in a gravitational field, and in the absence of other external fields, no torque is exerted in a test particle with only pole-dipole moments (no quadrupole or higher moments), as Eq. (22) shows; such particle is thus an ideal torque-free gyroscope. It is thus expected (being a gyroscope an object that *opposes* to changes in direction) to have a rotation axis fixed with respect to the Fermi-Walker transported tetrad, since the latter is by definition *locally* non-rotating. Actually the Fermi-Walker transported tetrad gets its physical significance as a locally non-rotating frame precisely by being anchored to local guiding gyroscopes [31] (defining the “local compass of inertia” [3]). *Only* if Mathisson-Pirani condition holds, is a gyroscope Fermi-Walker transported. Dixon condition yields a different equation, Eq. (C16) below, meaning that S^α undergoes transport orthogonal to P^α ; there is no conflict because the transport is along a different worldline, but the description is not as sound.

The Mathisson-Pirani condition may also provide the simplest description of the force / center of mass motion, and examples of that are the applications in Secs. IV B and V B, which we shall discuss next.

1. Comparison of the spin conditions in the applications of Sec. III

We start with the problem of fall along the axis in a Kerr spacetime discussed in Sec. VB.

Firstly, in order to compare the different spin conditions for the same problem, we must first establish how we ensure that we are dealing with the same particle. A pole dipole particle is characterized by its two moments: P^α and $S^{\alpha\beta}$; these are defined, cf. Eqs. (4)–(5), with respect to a point $z^\alpha(\tau)$ and an hypersurface of integration $\Sigma(u, \tau)$ (which may be thought as the instantaneous rest space of the observer measuring them); hence we prescribe the particle by giving these moments. Different representations of the same particle must yield the same moments with respect to the *same point* and observer. This is a crucial point made in [25].

We intend to describe what we naively expect, from symmetry arguments, to be a the fall along the z axis ($\theta = 0$ in Boyer-Lindquist coordinates) of a gyroscope whose initial position is some point along the z axis, and its initial velocity \vec{U} and spin \vec{S} are also along the axis of symmetry. It turns out, however, that such naive prescription of initial conditions not only does not completely determine the problem, nor does it ensure its axial symmetry. The reason is that in Relativity, and in contrast with Newtonian mechanics, the motion of the CM of an extended body is not fully determined by the force laws given its initial position and velocity; one needs a third condition, which is to prescribe the field of unit time-like vectors u^α relative to which the CM is computed (i.e., the field such that $S^{\alpha\beta}u_\beta = 0$), and which for an arbitrary choice breaks the axial symmetry. The momentum-velocity relation also depends on this choice, cf. Eq. (30), implying (hidden momentum) that \vec{U} will not in general be parallel to \vec{P} , so that they do not both lie along the z axis. Note that, as explained in Sec. IID, the acceleration of the CM does not originate solely from the force, but also from the variation of field u^α along the CM worldline.

In order to prescribe an axisymmetric problem, we tentatively set up as initial conditions $\vec{U}_{\text{in}} = U\vec{e}_z$ (by \vec{e}_z we denote, in Boyer-Lindquist coordinates, the basis vector $\vec{e}_r = \partial/\partial r$ along z), as initial position of the CM $x_{\text{CM}}^\alpha(u)$ some point in the z axis, and also some vector field u^α such that $\vec{u} = u\vec{e}_z$. In this case we expect the solution to be an axial fall.

The Pirani condition $S^{\alpha\beta}U_\beta = 0$, i.e., $u^\alpha = U^\alpha$ with this setup, *initially* satisfies the symmetry requirements, so let us start with it. In this case the momentum becomes (cf. Eq. (25))

$$P^\alpha = mU^\alpha - \epsilon^\alpha_{\beta\gamma\delta} S^\beta a^\gamma U^\delta, \quad (\text{C1})$$

and the spatial part of the equation of motion $F_G^\beta \equiv DP^\beta/d\tau = -\mathbb{H}^{\alpha\beta}S_\alpha$ (cf. Eq. (I.1b)) follows as:

$$m\vec{a} + \frac{D(\vec{S} \times_U \vec{u})}{d\tau} = \vec{F}_G = -\mathbb{H}^{i\alpha}S_\alpha \vec{e}_i \quad (\text{C2})$$

Initially, with $\mathbf{U}_{\text{in}} = U^0\vec{e}_0 + U^z\vec{e}_z$, one obtains $\mathbf{F}_{\text{Gin}} = -\mathbb{H}^{z\alpha}S_\alpha\vec{e}_z$ (it is straightforward check that along the axis we have $\mathbb{H}^{i\alpha} = 0$ if $i \neq z$); thus the force is along z as expected, which could have been inferred from symmetry arguments, given the axial symmetry of the initial setup and the fact that $\mathbb{H}_{\alpha\beta} \equiv R_{\alpha\mu\beta\nu}U^\mu U^\nu$ depends only on U^α . It is clear from the equation above that the most natural result, which is motion along the z axis, with the body accelerating in the same direction of the force (and of \vec{S}): $\vec{a} = a\vec{e}_z \Rightarrow \vec{S} \times_U \vec{a} = 0$, implying $P^\alpha = mU^\alpha$, and $F_G^\alpha = ma^\alpha$, is one²⁹ of the possible solutions of (C2). It is a “non-helical” solution since it is straightline, thus the description we seek. Hence we have solved our problem, the momentum-velocity relation having been naturally established in the course of the analysis.

Now let us compare with the equivalent descriptions for this problem given by other spin conditions. For a generic field u^α with \vec{u} not lying along the z axis, we no longer have axial symmetry and therefore we should not expect an axial fall as a solution. We know however from the analysis above with Pirani condition, that the axial fall indeed occurs given the appropriate initial conditions; so what we can hope for in the framework of a different spin condition, in general, is it to give a different (possibly exotic) but equivalent description of the same motion. However we would not know how to set it up, i.e., how to prescribe its initial conditions, without using the information extracted from the analysis above (it would not be clear that such motion was possible in the first place). Note that if one naively sets up $x_{\text{CM}}^\alpha(u)$ lying on the z axis, and then \vec{P} or \vec{U} (there is also an ambiguity on this choice, as they are not parallel, cf. Eq. (30)) along z , the solution of the equation of motion in general not will not be an axial fall, as it will not even be a solution equivalent to it up to a gauge transformation. We ensure that we are dealing with the same spinning particle, by imposing, as discussed above, P^α , and also $S^{\alpha\beta}$ *about* the same point $x_{\text{CM}}^\alpha(P)$, to be the same. Hence $\vec{P} = P\vec{e}_z$. In general, the condition $S^{\alpha\beta}u_\beta = 0$, for some family of unit time-like vectors u^α defined along the reference worldline, will define a centroid $x_{\text{CM}}^\alpha(u)$ at a different point compared to $x_{\text{CM}}^\alpha(U)$, not lying on the z axis, again manifesting the fact that the problem is no longer axisymmetric. Since U^α will not in general be parallel to P^α , cf. Eq. (30), the centroid also does not move parallel to the axis. Writing $S_{\alpha\beta} = \epsilon_{\alpha\beta\mu\nu}(S^u)^\mu u^\nu$, where $(S^u)^\alpha$ denotes the new spin vector, the force now reads

$$F_G^\alpha = -\frac{1}{2}R^\alpha_{\mu\nu\lambda}U^\mu S^{\nu\lambda} = -\star R^\alpha_{\mu\nu\lambda}U^\mu u^\nu (S^u)^\lambda, \quad (\text{C3})$$

²⁹ Other solutions were possible, this is due to the incompleteness of the gauge fixing of the Pirani condition. Note however that these are *not* equivalent solutions (not just a matter of choice between helical and non-helical representations), because, since U_{in}^α is fixed, they correspond to different P_{in}^α 's. They correspond to helical representations but of different solutions.

which depends both on U^α and u^α , and in general will not be also along the z axis. This clearly leads to a more complicated description of the same problem.

The case of the Dixon condition exemplifies some of these difficulties. Firstly we face the complicated equation relating P^α and U^α [24, 97]

$$U^\alpha = \frac{m}{M^2} \left(P^\alpha + \frac{2S^{\alpha\nu} R_{\nu\tau\kappa\lambda} S^{\kappa\lambda} P^\tau}{4M^2 + R_{\alpha\beta\gamma\delta} S^{\alpha\beta} S^{\gamma\delta}} \right) \quad (\text{C4})$$

which in general are not parallel; so if one wishes to study the problem of fall along z , we face the difficulties setting up initial conditions discussed above. For instance the ambiguity of which vector, \vec{U} or \vec{P} (if any) should be initially chosen parallel to z . Without using the knowledge extracted above with Pirani condition, and with Eq. (C4) coupled to (C3) it would be difficult to conclude that the axial fall occurs, and to what kind of solution it corresponds in this gauge — namely if it is or not an axisymmetric problem in this gauge. In order to represent the same spinning particle as above (which we know to represent the desired problem) it is \vec{P} that we must set along z , as discussed above. To find the force, given by Eq. (C3) above with $u^\alpha = P^\alpha/M$, one needs to know both³⁰ U^α and P^α (not only U^α , as with Pirani condition). It can then eventually be shown from (C4), as done in e.g. [98], even though not that easily, that for this setup, using Dixon condition we have $P^\alpha = mU^\alpha$, and therefore the solution coincides with the one obtained above with Pirani condition. Hence we end up precisely with the same solution, but taking a more complicated path.

In Sec. IV B 3 we concluded that in the equatorial plane of Kerr-dS, for suitable r and v^ϕ , spinning particles move in prograde circular geodesics. And we were able to do it only because we used Pirani condition. When this condition holds, the force is given by a contraction of the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$ with S^α , cf. Eq. (I.1 b); and from the analysis of the curvature scalar invariants we deduced that in the equatorial plane there is a velocity field Eq. (78) for which $\mathbb{H}_{\alpha\beta} = 0$; and that at some $r = r_{\text{geo}}$, solution of Eq. (89), that velocity corresponds to the velocity of the circular geodesic of a monopole particle. Along that circle, the equation of motion reduces to

$$\frac{DP^\alpha}{d\tau} = 0 \Leftrightarrow ma^\alpha - \epsilon^\alpha_{\beta\gamma\delta} U^\delta \frac{D(S^\beta a^\gamma)}{d\tau} = 0, \quad (\text{C5})$$

admitting $a^\alpha = 0$ as trivial solution, which is obviously a “non-helical” one; in that case the spinning particle will move along the circular geodesic. We would not be able to reach this conclusion using other spin conditions: for

a different condition $S^{\alpha\beta} u_\beta = 0$, $u^\beta \neq U^\alpha$ the force is no longer given by a contraction of the spin vector with the magnetic part of the Riemann tensor $\mathbb{H}_{\alpha\beta}$ (but instead by a rank two tensor $\mathcal{H}_{\alpha\beta} = \star R_{\alpha\mu\beta\nu} U^\mu u^\nu$ involving *both* u^β and U^β , cf. Eq. (C3)), and therefore a similar analysis in terms of scalar invariants of the Riemann tensor is not possible. In particular, starting with Dixon condition $S^{\alpha\beta} P_\beta = 0$, it would be virtually impossible, dealing the system formed by Eqs. (C9) and (9), coupled with the momentum-velocity relation (C4), to ever notice this effect.

As for the application in Sec. IV A, the motion in Schwarzschild spacetime of a particle with radial initial velocity, firstly notice that, contrary to what one might naively expect, we cannot extract from symmetry arguments much information about the force on a particle with generic spin S^α , as, due to it, the problem does not have spherical symmetry. Indeed, a force orthogonal to \vec{e}_r arises in the analogous electromagnetic setup, cf. Eq. (50). Using Pirani condition, with $\mathbf{U} = U^0 \vec{e}_0 + U^r \vec{e}_r$, we have, as discussed in Sec. IV A, $\mathbb{H}_{\alpha\beta} = 0 \Rightarrow DP^\alpha/d\tau = 0$. Hence again we have (C5) as equation of motion, with trivial solution $a^\alpha = 0 \Rightarrow P^\alpha = mU^\alpha$, i.e., the gyroscope radially moves along a geodesic. In the case of Dixon condition again we face Eq. (C4) relating P^α with U^α , not being clear which one to set initially along \vec{e}_r ; using the information extracted with Pirani condition, in order to represent the same problem, it must be \vec{P} ; then using Eq. (C14), and noticing that $(\mathbb{H}^P)_{\alpha\beta} = 0$, we see that, as expected $P^\alpha \parallel U^\alpha$ and $DP^\alpha/d\tau = 0$.

Now let us check the case of the electromagnetic analogous problem: magnetic dipole with initial radial velocity in the Coulomb field. Firstly we note that due to the (*gauge independent*) hidden momentum P_{hidEM}^α , in general P^α cannot be parallel to U^α ; furthermore, $F_{\text{EM}}^\alpha \neq 0$, and $a^\alpha \neq 0$; therefore, for the case of the Pirani condition, for a generic \vec{S} , and in the exact case, it is not trivial to prescribe the initial conditions for the non-helical motion we seek (although in this particular example, to first order in S , we can impose it by simply taking $S^{\alpha\beta} a_\beta \approx 0$). And in the case of the Dixon condition we face again a complicated equation relating P^α with U^α (and therefore F_{EM}^α with a^α), which is Eq. (35) of [29]. Hence, in the case of this application (not to compare with the gravitational case, but solely to solve the equations of motion), a condition analogous to the Corinaldesi-Papapetrou condition, $S^{\alpha\beta} u_\beta = 0$, with u^α corresponding to the static observers: $u^\alpha = \delta_0^\alpha \equiv \partial/\partial t$, seems the most appropriate: it makes the inertial hidden momentum vanish: $S^{\alpha\beta} Du_\beta/d\tau = 0$, leading to $P^\alpha = mU^\alpha + P_{\text{hidEM}}^\alpha$, which is the simplest momentum-velocity relation possible for this problem.

The bottom line is that dwelling on the relation between U^α and P^α is sometimes unnecessary work; in the absence of electromagnetic field, the difference between P^α and mU^α is *pure gauge*, as shown in Sec. II D. The inertial hidden momentum vanishes with a suitable choice

³⁰ It is for this reason that, although we can write, from Eq. (30), the expression $P^\alpha = M^2 U^\alpha/m + S^{\alpha\beta} (F_G)_{\beta}/m$ analogous to (C1), we cannot apply the same reasoning as with Pirani condition, because we only know in which direction the force points after solving (C4).

of the field u^α relative to which the CM is evaluated (i.e., choosing it parallel transported along the reference worldline, see Eq. (30)), leading to $P^\alpha \parallel U^\alpha$; this may actually be chosen as a spin supplementary condition, for a detailed discussion on it we refer to [40]. The spin condition is *gauge freedom*, and as such one should choose, in each application, the one that suits it most.

In the case of the applications presented, on top of the arguments above, we add the most important one in the context of this work, which is the fact that it is the Pirani condition that allows for the exact gravitoelectromagnetic analogies, and that it was thanks to that analogy that 1) in the application of Sec. VB, we understood the time component of the force, its fundamental difference with its electromagnetic counterpart, and were able to explain the spin-spin interaction energy with the absence of gravitational electromagnetic-like induction / the non-zero work done by the gravitomagnetic field; 2) in the case of the applications in Secs. IVB and IVA, we anticipated the existence of the circular geodesics for gyroscopes in the first case, and of the radial geodesics in the second case, which are related with very intrinsic differences between the two interactions that are made transparent in the symmetries of the tidal tensors (a formalism that *relies* on the Pirani condition). We believe this is enough to convince the reader that indeed, for these applications, the Pirani condition represents a clear advantage.

2. The difference between the two conditions for classical particles

We have seen in [25] that for a free spinning particle in flat spacetime the worldlines defined by Dixon condition (corresponding in this case to Mathisson non-helical solution) and all the Mathisson helices remain within a worldtube whose radius corresponds to the minimum size [41, 49] a classical spinning particle can have (if it is to have finite spin without violating special relativity). Let $S_\star^{\alpha\beta}$ denote the angular momentum taken about the center of mass $x_{\text{CM}}^\alpha(P)$ as measured in the $P^i = 0$ frame, i.e., $S_\star^{\alpha\beta} P_\beta = 0$; and let S_\star^α be the corresponding spin vector: $S_\star^{\alpha\beta} = \epsilon^{\alpha\beta}_{\mu\nu} S_\star^\mu P^\nu / M$. The radius of such worldtube is [25, 40, 41] $R_{\text{max}} = S_\star / M$. However, in the presence of strongly inhomogeneous external fields, the point we chose to represent the particle (i.e., where the fields are evaluated) makes a difference; moreover the force equation itself changes as we have seen in the previous section, cf. Eq. (C3). It is shown in [40], for the gravitational case (Kerr spacetime), that the solutions corresponding to different spin conditions diverge. Hence, depending on the supplementary conditions, we seemingly get different results. What this means however (this in agreement with the point of view in [40]) is the breakdown of the pole-dipole approximations (*not* that the spin conditions are not gauge after all!). The pole-dipole approximation is only acceptable when the choice of center of mass (and

the spin condition) does not matter; i.e., when the scale of variation of the external field is much larger than S/M , and if we follow the particle for a period of time that is short enough so that the many solutions do not diverge outside the worldtube.

Finally, there are a number of misconceptions which are usual in the literature regarding the effective differences between the two conditions, which we shall clarify. It is often asserted (in the absence of an electromagnetic field), that the two conditions are equivalent if terms quadratic in the spin are neglected [69, 81, 99], on the basis that, according to Eq. (C4), the difference U^α and P_β/M is of order $\mathcal{O}(S^2)$, which it is claimed to lead a difference between the mass dipole moments corresponding to each of the conditions, Eq. (13), of $S^{\alpha\beta} P_\beta/M - S^{\alpha\beta} U_\beta \sim \mathcal{O}(S^3)$. This is not correct; the analysis in [25], for the special case of flat spacetime, makes clear that the maximum distance between the worldline specified by $S_\star^{\alpha\beta} P_\beta = 0$ and the helical solutions allowed by $S^{\alpha\beta} U_\beta = 0$ is S_\star/M (i.e., $|(S^{\alpha\beta} P_\beta/M - S^{\alpha\beta} U_\beta)|_{\text{max}} = S_\star$). Also, the difference between mU^α and P^α is the hidden momentum; which, in the CM frame, reads $\vec{P}_{\text{hid}} = \vec{S} \times \vec{a}$ with Mathisson-Pirani condition, cf. Eq. (29). Thus in general it is not of order $\mathcal{O}(S^2)$ as stated in [81]; and not even necessarily of order $\mathcal{O}(S^1)$: for an helical solution, since $|\vec{a}| = \omega^2 R = Mv/S$ (see [25]), we have (in the CM frame) $|\vec{P} - m\vec{U}| = Mv$ which is of *zeroth order* in S . The origin of these misunderstandings is the fact that the momentum velocity relation depends on the spin condition, as is clear from Eq. (30); and in particular Eq. (C4), on which the analysis in [69, 81, 99] are based, *already assumes* the condition $S^{\alpha\beta} P_\beta = 0$.

We note also that it is not only when one compares an helical solution of $S^{\alpha\beta} U_\beta = 0$ and the worldline fixed by $S^{\alpha\beta} P_\beta = 0$ that the difference disagrees with the one usually mentioned in the literature; it also occurs between the latter and Mathisson-Pirani *non*-helical solution, if external fields are present. Let us see this with a concrete example. Consider first a magnetic dipole at rest, in the absence of any external fields. In this case, the *non-helical* solution allowed by Pirani's condition $S^{\alpha\beta} U_\beta = 0$ agrees with Dixon's center of mass worldline defined by $S^{\alpha\beta} P_\beta = 0$, since in this case $P^\alpha = MU^\alpha$, as shown by Eq. (C4). Thus, in this special case, both conditions pick the same reference worldline, i.e., a center of mass as determined in its own rest frame (a *proper* center of mass). Now bring a spherical charge close to the dipole, as depicted in Fig. 7. According to Pirani's condition, nothing changes, the center of mass $x_{\text{CM}}^\alpha(U)$ is still computed in the same frame. But it is not so with condition $S^{\alpha\beta} P_\beta = 0$, due to the hidden momentum term $\vec{P}_{\text{hidEM}} = \vec{\mu} \times_U \vec{E}$. This condition means that the center of mass $x_{\text{CM}}^\alpha(P)$ is computed in the zero 3-momentum frame ($\vec{P} = 0$), which is a frame moving, relative to the rest frame of $x_{\text{CM}}^\alpha(P)$ (the $\vec{U} = 0$ frame), in the direction along \vec{P}_{hidEM} . In order to see this, let us first

take the perspective of the $\vec{P} = 0$ frame; the vanishing of \vec{P} in this frame implies, via Eqs. (26) and (28), $\vec{U} = -\vec{P}_{\text{hidEM}}/m$; i.e., that $x_{\text{CM}}^\alpha(P)$ moves in this frame with 3-velocity $\vec{v} = \vec{U}/\gamma$, $\gamma \equiv -U_\alpha P^\alpha/M$. Therefore, the $\vec{P} = 0$ frame moves with velocity $-\vec{v}$ relative to the center of mass $x_{\text{CM}}^\alpha(P)$. This implies (see Fig. 1 of [25]) that $x_{\text{CM}}^\alpha(P)$ is shifted to the left relative to $x_{\text{CM}}^\alpha(U)$ (which is where $x_{\text{CM}}^\alpha(P)$ was in the absence of the electric field). Note that, since \vec{v} is constant, it follows from Eq. (27) of [25] that the two centers of mass $x_{\text{CM}}^\alpha(U)$ and $x_{\text{CM}}^\alpha(P)$ are points at rest with respect to each other (and to the charge in the apparatus of Fig. 7; i.e., they are at rest with respect to the static observers therein, according to the definition given in point 8 of Sec. ID). The shift $\Delta x^\alpha = x_{\text{CM}}^\alpha(P) - x_{\text{CM}}^\alpha(U)$ is given by Eq. (7) of [25] (setting therein $\bar{u}^\alpha = P^\alpha/M$; and Eq. (8) therein yields $-\Delta x^\alpha$). Thus, considering $\mu = \sigma S$, it is of order $\mathcal{O}(S^2)$.

a. The $\mathcal{O}(S^2)$ terms

We will now clarify the above discussion around the $\mathcal{O}(S^2)$ terms, and the reason why this order of magnitude is relevant. It is often asserted in the literature, in diverse contexts, that the $\mathcal{O}(S^2)$ contributions to P^α are negligible, so that in the purely gravitational case one may take $P^\alpha \approx mU^\alpha$ (which would imply, as explained above, that Mathisson *non-helical* solution would be identical to the worldline specified by $S^{\alpha\beta}P_\beta = 0$). Some authors [100] argue this is true in Post-Newtonian approximation (which is supported by our analysis in Sec. III); others argue it is so for “physical acceptable solutions” [69]; or that $\mathcal{O}(S^2)$ terms are of the order of magnitude of the quadrupole terms, which are to be neglected in the pole-dipole approximation [26]. But usually without a full, satisfactory justification (to our knowledge, such justification has not been put forth); that is our purpose in this section.

We will start by the gravitational case. We want to clarify if the second term in the expression for the momentum (25) is negligible in a pole-dipole approximation. We have already seen that for a generic spin condition (that is, for an arbitrary choice of centroid) it is not; an example are, as discussed above, Mathisson’s helical motions where this term is not negligible (nor of order $\mathcal{O}(S^2)$). In this analysis we will use the momentum velocity relation given by the Mathisson-Pirani condition, Eqs. (26)-(27), and consider the non-helical solution (in the purely gravitational case it is close to the worldline specified by $S^{\alpha\beta}P_\beta$ and the following analysis is similar for both). To dipole order, the only form of hidden momentum present in a purely gravitational system is the inertial one $P_{\text{hidI}}^\alpha = \epsilon_{\gamma\beta}^\alpha S^\beta a^\gamma$ (see Sec. IID). However, quadrupole (and higher order) terms contribute also to the hidden momentum. Contracting Eq. (142) with U_λ , yields an expression for the momentum, Eq. (26), accu-

rate to quadrupole order, embodying a quadrupole hidden momentum term P_{hidQ}^α :

$$P_{\text{hid}}^\alpha = \epsilon_{\gamma\beta}^\alpha S^\beta a^\gamma + P_{\text{hidQ}}^\alpha; \quad P_{\text{hidQ}}^\delta \equiv J^{\alpha\beta\gamma[\delta} R^{\sigma]}_{\gamma\alpha\beta} U_\sigma.$$

It is worth noting that P_{hidQ}^α , unlike the first term P_{hidI}^α , is not gauge (in general it cannot be made to vanish by a change of reference worldline). P_{hidQ}^α can be decomposed using Eqs. (5), leading to terms like $(m_Q)^{\gamma\delta}(\mathbb{E}^u)_{\gamma\delta}U^\alpha$, where $(m_Q)^{\gamma\delta}$ is the mass quadrupole defined in (149). Note also this important aspect: only the traceless part of $(m_Q)^{\gamma\delta}$ contributes in vacuum, since in that case $(\mathbb{E}^u)^\alpha_\alpha = 0$ (it is show using the above mentioned decomposition of $J_{\alpha\beta\gamma\delta}$ that the contribution of $(m_Q)^\gamma_\gamma$ to P_{hidQ}^α is zero).

We will exemplify the orders of magnitude of these hidden momentum terms using the application in Sec. III. For a non-helical solution, the acceleration comes in first approximation from the force F_G^α , i.e., the terms kept in (47), which are of order $\mathcal{O}(S)$. We will focus on the second term of (47), since it is dominant for large r (the analysis of the first term is analogous); the contribution P_{hidI}^α (i.e. $\vec{S} \times_U \vec{a}$) to the momentum is then typically:

$$|\vec{S} \times_U \vec{a}| \sim \frac{v}{m} |(E^u)_{ij}| S^2 \sim vm |(\mathbb{E}^u)_{ij}| R^2 \quad (\text{C6})$$

where we used $S \sim mv_{\text{rot}}R$, and allowing v_{rot} (the rotation velocity of a point at the body’s surface) to take its maximum value $v_{\text{rot}} = 1$; R is the radius of the spinning test body. Now, if no assumptions are made about the body’s multipole structure (such as, for instance, a close to “spherical” shape), the traceless part of $(m_Q)^{\gamma\delta}$ (which, as explained above, is the only part that contributes) must be allowed to be of the order $(m_Q)^{\gamma\delta} \sim mR^2$. That means that P_{hidQ}^α involves terms of the same order of magnitude of P_{hidI}^α :

$$|(m_Q)^{\gamma\delta}(\mathbb{E}^u)_{\gamma\delta}v^i| \sim vm |(\mathbb{E}^u)_{ij}| R^2 \sim |\vec{S} \times_U \vec{a}|.$$

This justifies the neglect of $\mathcal{O}(S^2)$ hidden momentum terms in the pole-dipole scheme, and the claim in [26], *if no assumptions are made* about the body’s multipole structure (i.e., the only assumption made about the body is that it be small enough so that higher order multipoles are negligible). On the other hand, if one assumes that the body is nearly “spherical”, so that its traceless quadrupole is very small, which is a reasonable assumption, then it will make sense to keep these $\mathcal{O}(S^2)$ hidden momentum terms in a pole-dipole approximation. This means that the treatments in [24], studying the solution given by Dixon-Tulczyjew condition without making $P^\alpha \approx mU^\alpha$, and in [40], comparing with the condition $P^\alpha \parallel U^\alpha$ (the difference between them being $\mathcal{O}(S^2)$); or the work in [100] comparing Mathisson non-helical solution with Dixon-Tulczyjew, are not a waste of time and make sense indeed.

In the electromagnetic case the situation is different. As explained above, in general $P_{\text{hidEM}}^\alpha \neq 0$, so that

to dipole order one does not have $P^\alpha \approx mU^\alpha$ (nor that Mathisson's non-helical solution is approximated by Dixon's Tulczyjew). As for P_{hidI}^α , it cannot also be neglected by arguments in the line of the ones above, and should in principle be kept, because when the particle has a net charge, the Lorentz force, first term, of Eq. (39), will yield a contribution to the acceleration independent of $\vec{\mu}$ (or \vec{S}), hence $\vec{S} \times_U \vec{a}$ will be of first order in S . If, on the other hand, one considers the magnetic moment $\vec{\mu}$ to arise from an uncharged body (for instance, the current loop depicted in Fig. 3a), then P_{hidI}^α is typically (again we focus on the dominant first term of (46)):

$$|\vec{S} \times_U \vec{a}| \sim \frac{v\sigma}{m} |(E^u)_{ij}| S^2 \sim m\sigma v |(E^u)_{ij}| R^2 \sim vm\sigma\phi \frac{R^2}{r^2}. \quad (\text{C7})$$

To get an estimate of the quadrupole contribution to the hidden momentum, we contract Eq. (109) with U_β , yielding $2m^{[\alpha}{}_{\rho\mu} F^{\beta]\mu;\rho} U_\beta$ which involves terms of the type

$$(h^U)_\sigma^\alpha \mathcal{J}^{\delta\gamma\sigma} 2E_{\delta\gamma} \sim \sigma m |(E^u)_{ij}| R^2 \sim |\vec{S} \times_U \vec{a}|/v \quad (\text{C8})$$

($\mathcal{J}^{\alpha\beta\gamma}$ is defined in (112); in this case we base our estimate in the moments of the spatial current, i.e., $(h^U)_\sigma^\gamma \mathcal{J}^{\delta\gamma\sigma}$) where we used $(h^U)_\sigma^\alpha \mathcal{J}^{\delta\gamma\sigma} \sim \mu R \sim \sigma m R^2$. Hence, *for this special case* (the one considered in Sec. III), arguments analogous to the given above for the gravitational case apply.

3. Impact of the Dixon-Tulczyjew condition on the gravito-electromagnetic analogies

a. Analogy based on tidal tensors

If one uses Dixon-Tulczyjew condition, then $S^{\mu\nu} = \epsilon^{\mu\nu\tau\lambda} (S^P)_\tau P_\lambda / M$ (with Pirani's condition, it would be $S^{\mu\nu} = \epsilon^{\mu\nu\tau\lambda} S_\tau U_\lambda$), and therefore Eq. (I.1b) of Table I would read instead:

$$\frac{DP^\alpha}{d\tau} = -\frac{1}{2} R^\alpha_{\beta\mu\nu} S^{\mu\nu} U^\beta \equiv -\mathcal{H}^{\gamma\alpha} (S^P)_\gamma, \quad (\text{C9})$$

where $\mathcal{H}_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\nu} U^\nu P^\mu / M$. Thus the force is still of course given by a contraction of a rank 2 tensor $\mathcal{H}_{\alpha\beta}$ with the spin 4-vector S^α ; however, this new “tidal tensor” does not coincide with the magnetic part of the Riemann tensor $(\mathbb{H}^u)_{\alpha\beta} = \star R_{\alpha\mu\beta\nu} u^\mu u^\nu$ as measured by some observer u^α , because $\mathcal{H}^\alpha_\gamma$ results from a contraction with two different vectors (P^α and U^α). The analogy in Table 1 will no longer be exact, since by taking the trace and antisymmetric parts of $\mathcal{H}_{\alpha\beta}$, one does not obtain equations of the type (I.2b)-(I.3b). The equations one will obtain will no longer be projections of Einstein field equations, and they will no longer be analogous to their gravito-electric counterparts Eqs. (3b)-(4b) of Table 1 of [57], nor to their electromagnetic counterparts (3a)-(6a) of [57], since no longer will they be equations involving only tidal tensors and sources. Also, the tensorial structure of $\mathcal{H}_{\alpha\beta}$

(unlike $\mathbb{H}_{\alpha\beta}$) is not similar to its gravito-electric counterpart, i.e., it is not spatial, nor it has to be symmetric in vacuum.

The reason for all this being, as stated above, the fact that $\mathcal{H}_{\alpha\beta}$ (i.e., Papapetrou Eq. (C9) above) involves *both* U^α and P^α , so one cannot save the (exact) analogy by, e.g., projecting in terms of P^α/M instead of U^α . Let us compute explicitly the deviation from the analogue form. Firstly we rewrite Eq. (C4) as:

$$U^\alpha = U_\parallel^\alpha + U_\perp^\alpha; \quad U_\parallel^\alpha \equiv \frac{m}{M^2} P^\alpha; \quad U_\perp^\alpha \equiv (h^P)_\beta^\alpha P^\beta \quad (\text{C10})$$

where U_\parallel^α and U_\perp^α are, respectively, the projections of U^α parallel and orthogonal to P^α . We may then write $\mathcal{H}_{\alpha\beta}$ in the two alternative forms:

$$\mathcal{H}_{\alpha\beta} = \frac{m}{M} \mathbb{H}_{\alpha\beta} + \frac{1}{M} \star R_{\alpha\mu\beta\nu} P^\mu U^\nu \quad (\text{C11})$$

$$= \frac{m}{M} (\mathbb{H}^P)_{\alpha\beta} + \frac{1}{M} \star R_{\alpha\mu\beta\nu} P^\mu U_\perp^\nu; \quad (\text{C12})$$

note that

$$P_{\text{hid}}^\alpha = -\frac{M^2}{m} (h^U)_\beta^\alpha U_\perp^\beta. \quad (\text{C13})$$

If $F^{\alpha\beta} = 0$, U_\perp^α is the second term of (C4); it is useful to write it in terms of tidal tensors:

$$U_\perp^\alpha = \frac{m}{M^2} \frac{\epsilon^{\alpha\gamma}{}_{\tau\delta} (S^P)^\tau P^\delta (\mathbb{H}^P)_{\sigma\gamma} (S^P)^\sigma}{M^2 + (\mathbb{F}^P)^{\lambda\sigma} (S^P)_\lambda (S^P)_\sigma}. \quad (\text{C14})$$

where $(\mathbb{H}^P)_{\alpha\gamma} \equiv \star R_{\alpha\beta\gamma\delta} P^\beta P^\delta / M^2$ and $(\mathbb{F}^P)_{\alpha\gamma} \equiv \star R \star_{\alpha\beta\gamma\delta} P^\beta P^\delta / M^2$ denote, respectively, the gravito-magnetic tidal tensor and the “ \mathbb{F} ” tensor [34, 101] measured by an observer of 4-velocity $u^\alpha = P^\alpha/M$. Thus we see that indeed the analogy in Table I is not exact if one uses the spin condition $S^{\alpha\beta} P_\beta = 0$; the differing terms being $\mathcal{O}(S^2)$ if one considers the purely gravitational case ($F^{\alpha\beta} = 0$), as shown by Eqs (C11)-(C14), or $\mathcal{O}(S)$ if one considers the gravitational force F_G^α in the general case that the two interactions are present, and assuming $\mu^{\alpha\beta} = \sigma S^{\alpha\beta}$, since in this case $P_{\text{hid}}^\alpha \sim \mathcal{O}(S)$ due to P_{hidEM}^α , cf. Eq. (25) (for an explicit expression for P_{hid}^α under Dixon-Tulczyjew condition, see Eqs. (35) and (70) of [29]).

b. Analogy based on GEM inertial fields

The expressions for the momentum (29), force [5, 34], and spin evolution (24), obtained within the gravito-electromagnetic analogy based in the 1+3 formalism, discussed in [34], also rely on the condition $S^{\alpha\beta} U_\beta = 0$.

Hidden momentum. — If the reference worldline is the one given by condition $S^{\alpha\beta} P_\beta = 0$, the momentum no longer takes the forms (27)-(29). The inertial hidden momentum $P_{\text{hidI}}^\alpha = U_\beta D S^{\beta\alpha} / d\tau$ in (25) is no longer $S^{\alpha\beta} a_\beta$, thus no longer exhibits an exact formal analogy

with its electromagnetic counterpart $P_{\text{hidEM}}^\alpha = -\mu^{\alpha\beta} E_\beta$. The analogy can however be recovered at an approximate level [29]. Using $P^\alpha = mU^\alpha + P_{\text{hid}}^\alpha$ and $S^{\alpha\beta} P_\beta = 0$ we can write:

$$P_{\text{hidI}}^\alpha = S^{\alpha\beta} a_\beta + \frac{1}{m} \left[\frac{dm}{d\tau} S_{\beta\perp}^\alpha U^\beta + \frac{D}{d\tau} (S_{\beta\perp}^\alpha P_{\text{hid}}^\beta) \right] \quad (\text{C15})$$

i.e., $P_{\text{hidI}}^\alpha = S^{\alpha\beta} a_\beta + \mathcal{O}(S^n)$, being $n = 3$ if $F^{\alpha\beta} = 0$ (and there are no other external forces), as follows from Eqs. (C13)-(C14), or $n = 2$ if $F^{\alpha\beta} \neq 0$, as follows from Eqs. (25) and (C13). Hence, at least to first order in S , the analogy holds.

Force on the gyroscope. — The force Eq. (5) of [5], when it holds, is the same as (I.1b), only written in a different formalism; hence the discussion in Sec. C 3 applies.

Spin precession. — As discussed above, only if Pirani's condition holds, a gyroscope (in the absence of electromagnetic field) undergoes Fermi-Walker transport. If one adopts the condition $S^{\alpha\beta} P_\beta = 0$, we have (cf. Eq. (7.11) of [22]):

$$\frac{DS^\alpha}{d\tau} = S_\sigma \mathcal{U}^\alpha \frac{DU^\sigma}{d\tau} = \frac{1}{M^2} P^\alpha S_\sigma \frac{DP^\sigma}{d\tau} \quad (\text{C16})$$

($\mathcal{U}^\alpha \equiv P^\alpha/M$) which is a different transport law, preserving the orthogonality to P^α (instead of U^α), dubbed in [22] ("M transport"). Note that there is no conflict with Eq. (23) because the transport is along a different worldline. In the CM frame we have

$$\frac{DS^i}{d\tau} = \frac{1}{M^2} P_{\text{hid}}^i S_\sigma \frac{DP^\sigma}{d\tau}.$$

Thus, in the comoving (congruence adapted: $\vec{\Omega} = \vec{\omega}$, see [34]) frame $\mathbf{e}_{\hat{\alpha}}$, the expression corresponding to (24) (with $\vec{B} = 0$) reads with this spin condition

$$\frac{d\vec{S}}{d\tau} = \frac{1}{2} \vec{S} \times \vec{H} + \frac{1}{M^2} \vec{P}_{\text{hid}} S_\sigma \frac{DP^\sigma}{d\tau} \quad (\text{C17})$$

showing that the term breaking the analogy with the electromagnetic counterpart comes from the hidden momentum. In the absence of electromagnetic field and other external forces and torques, $P_{\text{hid}}^\alpha \sim \mathcal{O}(S^2)$, cf. Eqs. (C13)-(C14), and thus the extra term is of order $\mathcal{O}(S^3)$. If the electromagnetic field is present (and $\mu^{\alpha\beta} = \sigma S^{\alpha\beta}$), $P_{\text{hid}}^\alpha \sim \mathcal{O}(S)$ and the extra term is of order $\mathcal{O}(S^2)$, cf. Eq. (25).

4. Conserved quantities with Dixon-Tulczyjew condition and electromagnetic induction

Let us start by the purely gravitational case. With Dixon condition, it is in the frame of zero 3-momentum ($P^i = 0$) that the time projection of the force on a gyroscope vanishes:

$$\frac{DP^\alpha}{d\tau} \mathcal{U}_\alpha = -\frac{dM}{d\tau} = -\mathcal{H}^{\gamma\alpha} (S^P)_\gamma \mathcal{U}_\alpha = 0$$

($\mathcal{U}^\alpha \equiv P^\alpha/M$; $M^2 \equiv P^\alpha P_\alpha$), the last equality being obtained using Eqs. (C12) and (C14). M (not m) is thus the conserved quantity, and has the interpretation of energy of the body relative to the $P^i = 0$ frame.

In the electromagnetic case, from Eq. (8), we have, for a magnetic dipole:

$$-\frac{dM}{d\tau} = \frac{DP^\alpha}{d\tau} \mathcal{U}_\alpha = \frac{1}{2} F_{\mu\nu}{}^{;\alpha} \mathcal{U}_\alpha \mu^{\mu\nu}$$

which is non zero if the derivative of $F^{\alpha\beta}$ is not zero along P^α . Therefore, in this respect, the conclusions are not very different comparing with Pirani condition: with the latter m is conserved for a gyroscope in a gravitational field, and in the electromagnetic counterpart its variation reflects the phenomenon of electromagnetic induction; with Dixon's condition, we have that M is conserved in the gravitational case, and that it varies in the electromagnetic case, again due to induction effects (only this time they are measured relative to the $P^i = 0$ frame).

Appendix D: Electromagnetic momentum and hidden momentum

Let $(T_{\text{tot}})^{\alpha\beta} = (T_{\text{matter}})^{\alpha\beta} + \Theta^{\alpha\beta}$ be the total energy momentum of the system, with $(T_{\text{matter}})^{\alpha\beta}$ being the energy-momentum of the matter and $\Theta^{\alpha\beta}$ the electromagnetic stress-energy tensor:

$$\Theta^{\alpha\beta} = \frac{1}{4\pi} \left[(F_{\text{tot}})^{\alpha\gamma} (F_{\text{tot}})^\beta{}_\gamma - \frac{1}{4} g^{\alpha\beta} (F_{\text{tot}})_{\mu\nu} (F_{\text{tot}})^{\mu\nu} \right] \quad (\text{D1})$$

$(T_{\text{tot}})^{\alpha\beta}$ obeys the conservation equations: $(T_{\text{tot}})^{\alpha\beta}{}_{;\beta} = 0$; and $\Theta^{\alpha\beta}{}_{;\beta} = -(F_{\text{tot}})^{\alpha\beta} j_\beta$ (cf. Eq. (7.36) of [70], Eq. (12.118) of [63]). Therefore

$$(T_{\text{tot}})^{\alpha\beta}{}_{;\beta} = 0 \Rightarrow (T_{\text{matter}})^{\alpha\beta}{}_{;\beta} = -\Theta^{\alpha\beta}{}_{;\beta} = F^{\alpha\beta} j_\beta \quad (\text{D2})$$

The electromagnetic stress-energy tensor splits in three parts (Eq. (36) of [39]):

$$\Theta^{\alpha\beta} = (\Theta_{\text{ext}})^{\alpha\beta} + (\Theta_{\times})^{\alpha\beta} + (\Theta_{\text{self}})^{\alpha\beta}$$

where

$$(\Theta_{\text{ext}})^{\alpha\beta} = \frac{1}{4\pi} \left[F^{\alpha\gamma} F^\beta{}_\gamma - \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right]$$

$$(\Theta_{\times})^{\alpha\beta} = \frac{1}{4\pi} \left[F^{\alpha\gamma} (F_p)^\beta{}_\gamma + (F_p)^{\alpha\gamma} F^\beta{}_\gamma - \frac{1}{2} g^{\alpha\beta} (F_p)_{\mu\nu} F^{\mu\nu} \right]$$

$$(\Theta_{\text{self}})^{\alpha\beta} = \frac{1}{4\pi} \left[(F_p)^{\alpha\gamma} (F_p)_{\gamma}^{\beta} - \frac{1}{4} g^{\alpha\beta} (F_p)_{\mu\nu} (F_p)^{\mu\nu} \right].$$

Since we are considering test particles, strictly speaking, one was supposed to neglect the electromagnetic field $(F_p)^{\alpha\gamma}$ produced by the test particle. However that would be inconsistent with taking into account its dipole structure, as the cross term $(\Theta_{\times})^{\alpha\beta}$ above yields, as we shall see, a contribution to the momentum of the type $\vec{\mu} \times \vec{E}$; this cannot be consistently neglected in an approximation where couplings of the type $\vec{B} \cdot \vec{\mu}$ are taken into account. This is in agreement with the treatment in Sec. VB of [29].

The part of $\Theta^{\alpha\beta}$ which is to be neglected in this approximation is the term $(\Theta_{\text{self}})^{\alpha\beta}$, which is quadratic in the test particle's field, and is related with the electromagnetic self force [39]. Hence, in this paper:

$$\Theta^{\alpha\beta} \simeq (\Theta_{\text{ext}})^{\alpha\beta} + (\Theta_{\times})^{\alpha\beta}.$$

If the only currents present are from the internal structure of the test particle: $j^{\alpha} = (j_p)^{\alpha}$, then $(F_{\text{ext}})^{\alpha\beta}_{;\beta} = 0$, and therefore $(\Theta_{\text{ext}})^{\alpha\beta}_{;\beta} = 0$. That is: the stress energy tensor of the external electromagnetic field is independently conserved (cf. [39] pp 7-8). Also, if the only matter present is the test particle's (herein we are interested in a system consisting solely on a test particle plus an electromagnetic field): $(T_{\text{matter}})^{\alpha\beta} = T^{\alpha\beta}$ ($T^{\alpha\beta}$ denotes the *energy momentum of the test particle*); then it follows from (D2)

$$T^{\alpha\beta}_{;\beta} = -\Theta^{\alpha\beta}_{;\beta} = -(\Theta_{\times})^{\alpha\beta}_{;\beta} = F^{\alpha\beta} (j_p)_{\beta} \quad (\text{D3})$$

which is Dixon's equation (1.1) of [22] (with a relative minus due to the different signature), Eq. (5) of [102], or Eq. (10) of [29] (with $f_a = 0$).

By obeying (D3), $T^{\alpha\beta}$ encodes the hidden momenta of the particle. In order to see that, consider for simplicity flat spacetime and Lorentz coordinates. In this case we have $T^{\alpha\beta}_{;\beta} = F^{\alpha\beta} j_{\beta}$. Noting that $T^{\alpha\gamma} = (T^{\alpha\beta} x^{\gamma})_{;\beta} - T^{\alpha\beta}_{;\beta} x^{\gamma}$, we have, from Eq. (4) (work in the frame comoving with the test particle):

$$\begin{aligned} P^i &\equiv \int T^{0i} d^3x \\ &= \int (T^{00} x^i)_{,0} d^3x + \int (T^{0k} x^i)_{,k} d^3x - \int F^{0k} j_k x^i d^3x \\ &= \int (T^{00} x^i)_{,0} d^3x + \oint_S T^{0k} x^i n_k dS - \int F^{0k} j_k x^i d^3x. \end{aligned}$$

For a stationary configuration, the first term vanishes; the surface integral in second term also vanishes in a surface enclosing the test particle. Hence, neglecting the variation of $F^{\alpha\beta}$ along the dipole, and using definitions (7) and (10), we have

$$P^i = -F^{0k} \int j_k x^i d^3x = -F^{0\beta} \mu^i_{\beta} = E^{\beta} \mu^i_{\beta}. \quad (\text{D4})$$

The second equality holds in stationary conditions (so that $\int j^i x^k d^3x = \int j^{[i} x^{k]} d^3x = \mu^{ki}$ by conservation of charge) which is the problem at hand. This equals P^i_{hidEM} [last term of Eqs. (25) or (27)], as expected, since for a stationary configuration $U^i = a^i = 0$ and therefore

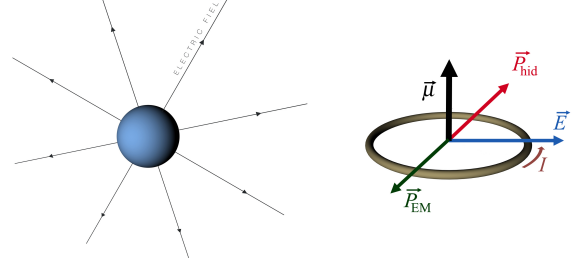


Figure 7: A magnetic dipole subject to the Coulomb field of a spherical charge. Although the dipole is not moving, it has a non-vanishing hidden momentum $\vec{P}_{\text{hid}} = \vec{P}_{\text{hidEM}} = \vec{\mu} \times \vec{E}$ which exactly cancels out the electromagnetic field momentum $\vec{P}_{\text{EM}} = -\vec{\mu} \times \vec{E}$, as one concludes from the conservation equations $T^{\alpha\beta}_{;\beta} = 0$.

$P^i = P^i_{\text{hid}} = P^i_{\text{hidEM}}$, cf. Eqs. (26), (28). In vector notation, Eq. (D4) reads $\vec{P} = \vec{\mu} \times \vec{E}$; this is precisely *minus* the electromagnetic momentum generated by the presence of a magnetic dipole in an external electromagnetic field, as is easy to show. In order to see that, first note that such momentum reduces to the cross term $P_{\times}^{\alpha} \equiv \int_{\Sigma(u)} (\Theta_{\times})^{\alpha\beta} d\Sigma_{\beta}$, as $(\Theta_{\text{self}})^{\alpha\beta}$ is to be neglected in accordance with discussion above. Also one notes that, in the particle's frame ($U^i = 0$), the electric field produced by the magnetic dipole is zero; therefore, the spatial momentum is given by (see e.g. [63] p. 286, [59]):

$$\vec{P}_{\times} = \frac{1}{4\pi} \int \vec{E} \times \vec{B}_{\text{dipole}} d^3x = \vec{E} \times \vec{\mu}, \quad (\text{D5})$$

i.e., the volume integral of the Poynting vector $\vec{p} = \vec{E} \times \vec{B}_{\text{dipole}}/4\pi$. In order to convince ourselves that the existence of the hidden momentum $\vec{P}_{\text{hid}} = -\vec{P}_{\times}$ is a requirement of the conservation equations, consider the simple example depicted in Fig. 7, a magnetic dipole in the Coulomb field of a spherical charge. Although the configuration is stationary (from the viewpoint of the static observers), the dipole possesses a non-vanishing spatial momentum, since it follows from equation $(T_{\text{tot}})^{\alpha\beta}_{;\beta} = 0$ that, for a stationary configuration, $P^i_{\text{tot}} = 0$. Since $P^{\alpha}_{\text{tot}} \equiv P^{\alpha}_{\text{matter}} + P^{\alpha}_{\text{EM}}$, then $P^i_{\text{matter}} = -P^i_{\text{EM}}$. In this case, P^i_{\times} given by (D5) is the total electromagnetic momentum: $P^i_{\times} = P^i_{\text{EM}}$. Thus we know that the matter must possess a hidden momentum $\vec{P}_{\text{matter}} = -\vec{P}_{\text{EM}} = \vec{\mu} \times \vec{E}$; and this momentum must be carried by the current loop, since the static charge can carry no momentum.

Thus, we succeeded in showing that in stationary conditions, the conservation equations imply $-P^{\alpha}_{\text{EM}} = P^{\alpha}_{\text{hidEM}}$. For more general conditions we cannot follow the same procedure to obtain such equality (see also e.g. [29],

p. 20). But from Eq. (27) we can see that indeed Dixon's equations (which come precisely from the conservation Eq. (D3)) imply that the equality always holds.

1. Physical model of electromagnetic hidden momentum. Inertial analogue.

In this work we dubbed the contribution P_{hid}^α to P_{hidEM}^α as hidden momentum “of electromagnetic origin”, because it only exists when an electromagnetic field is present; as explained above, for stationary configurations it equals *minus* the momentum of the field. It is however purely mechanical in nature [29, 62, 103], and can be explained by different models (see [103] p. 520, [59, 62]), all yielding the same results. For our purposes it suffices to consider the more simple and intuitive one, given in Fig. 9 of [62]: a square current loop, under the action of an electric field (which, to this accuracy, is taken constant along the body). The electric field pointing upwards, accelerates the charges on the right segment, and decelerates the ones on the left segment; thereby causing an increase in the concentration of positive charges of in bottom segment (*not* the upper segment, as one might naively expect!). That can be seen recalling that the flux of particles nv (i.e., the number of particles per unit time) is *constant* throughout the loop (otherwise, charge would be piling up somewhere), i.e., $n_{\text{bot}}v_{\text{bot}} = v_{\text{top}}n_{\text{top}}$; therefore, if $v_{\text{top}} > v_{\text{bot}}$, then $n_{\text{bot}} > n_{\text{top}}$. n is equal for the left and right segments. Since the relativistic density of momentum is $p = nv\gamma m$, non-relativistically ($\gamma \approx 1$), the total momentum of the loop would be zero, as the momenta from the top and bottom, and the left and right segments, would cancel out. In the relativistic case, however, due to γ , the top segment has larger momentum than the bottom one; which means that the loop, in spite of being at rest, has a non-vanishing momentum [62, 103] $P_{\text{hid}} = m(\gamma_{\text{top}} - \gamma_{\text{bot}})lI/q$ (note that $lI/q = N_{\text{top}}v_{\text{top}} = N_{\text{bot}}v_{\text{bot}}$, with $N \equiv$ number of charges in a segment). That is, hidden momentum. Since $m(\gamma_{\text{top}} - \gamma_{\text{bot}}) = qEw$ (i.e., the gain in kinetic energy going from the bottom to the top is the work done by the electric field), and $\mu = Iwl$, we obtain, for the setup in Fig. 9 of [62], $\vec{P}_{\text{hidEM}} = \vec{\mu} \times \vec{E}$.

Inertial analogue.— Replacing the charges in the loop

of Fig. 9 of [62] by masses, and the electric field \vec{E} by the gravitoelectric field $\vec{G} = -\vec{a}$, as measured in the frame comoving with the CM (i.e., \vec{a} is the CM acceleration), one obtains analogously a hidden momentum $\vec{S} \times \vec{G}$, in this case originated by the inertial force $\vec{G} = -\vec{a}$. This observation makes natural the similarity between the “electromagnetic” hidden momentum $\vec{\mu} \times \vec{E}$ (i.e., $\mu^{\theta\alpha}E_\theta$) and the term $\vec{a} \times \vec{S}$ (i.e., $S^{\alpha\theta}a_\theta$) coming from the Mathisson-Pirani condition. However it must be noted that this model can *only* be used to interpret the hidden momentum term given by Mathisson-Pirani condition; i.e., when the CM is computed in its rest frame. For any other spin condition/choice of observers u^α relative to which the CM is evaluated, $P_{\text{hid}}^\alpha \neq S^{\alpha\theta}a_\theta$ (in general it is not related to the acceleration of the body); also, consistently, the CM $x_{\text{CM}}^\alpha(u)$ (thus the apparatus in Fig. 9 of [62]) is not at rest in the frame $u^i = 0$. This is the case even if one computes the CM relative to an observer u^α momentarily comoving with the proper CM $x_{\text{CM}}^\alpha(U)$, but with different acceleration: the CM is measured to be at the same position $x_{\text{CM}}^\alpha(u) = x_{\text{CM}}^\alpha(U)$, but their velocities are different, as they depend on the acceleration, see Eq. (27) of [25].

Hence, albeit the analogue model in Fig. may be used to interpret the term P_{hidI}^α and the reason for its formal similarity with P_{hidEM}^α , one must bear in mind that the fundamental, generic, explanation of the inertial hidden momentum is the one given in Sec. IID above, and Sec. VI of [25]: $x_{\text{CM}}^\alpha(u)$ depends on u^α ; if u^α changes along the curve, the CM velocity changes in a non-trivial way, causing it to move in the $P^i = 0$ frame; conversely, in the CM frame $U^i = 0$, P^i is not zero, and that is hidden momentum.

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